

POINTWISE SECOND-ORDER NECESSARY CONDITIONS FOR STOCHASTIC OPTIMAL CONTROLS, PART II: THE GENERAL CASE*

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Abstract. This paper is the second part of our series of work to establish pointwise second-order necessary conditions for stochastic optimal controls. In this part, we consider the general cases, i.e., the control region is allowed to be nonconvex, and the control variable enters into both the drift and the diffusion terms of the control systems. By introducing four variational equations and four adjoint equations, we obtain the desired necessary conditions for stochastic singular optimal controls in the sense of Pontryagin-type maximum principle.

Key words. Stochastic optimal control, needle variation, pointwise second-order necessary condition, variational equation, adjoint equation.

AMS subject classifications. Primary 93E20; Secondary 60H07, 60H10.

1. Introduction. Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space (satisfying the usual conditions), on which a 1-dimensional standard Wiener process $W(\cdot)$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration generated by $W(\cdot)$ (augmented by all of the P -null sets).

We consider the following controlled stochastic differential equation

$$(1.1) \quad \begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), & t \in [0, T], \\ x(0) = x_0, \end{cases}$$

with a cost functional

$$(1.2) \quad J(u(\cdot)) = \mathbb{E} \left[\int_0^T f(t, x(t), u(t))dt + h(x(T)) \right].$$

Here $u(\cdot)$ is the control variable valued in a set $U \subset \mathbb{R}^m$ (for some $m \in \mathbb{N}$), $x(\cdot)$ is the state variable with values in \mathbb{R}^n , and $b, \sigma : \Omega \times [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ (for some $n \in \mathbb{N}$), $f : \Omega \times [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ and $h : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ are given functions (satisfying some conditions to be given later). As usual, for maps $\varphi = b, \sigma, f$, denote by $\varphi_x(\omega, t, x, u)$, $\varphi_{xx}(\omega, t, x, u)$, $\varphi_{xxx}(\omega, t, x, u)$ and $\varphi_{xxxx}(\omega, t, x, u)$ its first, second, third and forth order partial derivatives with respect to the variable x at (ω, t, x, u) , respectively. And, when the context is clear, we omit the $\omega(\in \Omega)$ argument in the defined functions.

Denote by $\mathcal{B}(\mathcal{X})$ the Borel σ -field of a metric space \mathcal{X} , and by \mathcal{U}_{ad} the set of $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable and \mathbb{F} -adapted stochastic processes valued in U . Any $u(\cdot) \in \mathcal{U}_{ad}$ is called an admissible control. The stochastic optimal control problem considered in this paper is to find a control $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ such that

$$(1.3) \quad J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}} J(u(\cdot)).$$

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Any $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ satisfying (1.3) is called an optimal control. The corresponding state $\bar{x}(\cdot) (= \bar{x}(\cdot; x_0, \bar{u}(\cdot)))$ to (1.1) is called an optimal state, and $(\bar{x}(\cdot), \bar{u}(\cdot))$ is called an optimal pair.

One of the central problems in stochastic control theory is to derive necessary conditions for the optimal pair $(\bar{x}(\cdot), \bar{u}(\cdot))$. Before analyzing this issue in detail, we recall first some elementary facts from the classical calculus. Let us consider a minimizer $x_0 (\in G)$ of a smooth function $f(\cdot)$ defined on a set $G \subset \mathbb{R}^n$, i.e., x_0 satisfies

$$(1.4) \quad f(x_0) = \inf_{x \in G} f(x).$$

If a nonzero vector $\ell \in \mathbb{R}^n$ is admissible (i.e., there is a $\delta > 0$ so that $x_0 + s\ell \in G$ for any $s \in [0, \delta]$), then one has the following first-order necessary condition:

$$(1.5) \quad 0 \leq \lim_{s \rightarrow 0^+} \frac{f(x_0 + s\ell) - f(x_0)}{s} = \langle f_x(x_0), \ell \rangle.$$

When $\langle f_x(x_0), \ell \rangle = 0$ holds, i.e., (1.5) degenerates, then one can obtain further a second-order necessary condition as follows:

$$(1.6) \quad 0 \leq 2 \lim_{s \rightarrow 0^+} \frac{f(x_0 + s\ell) - f(x_0)}{s^2} = \langle f_{xx}(x_0)\ell, \ell \rangle.$$

In the particular case that G is convex, by (1.5), one has

$$(1.7) \quad 0 \leq \langle f_x(x_0), x - x_0 \rangle, \quad \forall x \in G.$$

When $f_x(x_0) = 0$, then it follows from (1.6) that

$$(1.8) \quad 0 \leq \langle f_{xx}(x_0)(x - x_0), x - x_0 \rangle, \quad \forall x \in G.$$

Clearly, compared to the first-order necessary condition (1.5)/(1.7), the second-order necessary condition (1.6)/(1.8) can be used to single out the possible minimizer x_0 from a smaller subset of G . From the above analysis on the minimization problem (1.4), it is easy to see the following:

- 1) Usually, one has to impose more regularity on the data (say C^2 for $f(\cdot)$) for the second-order necessary condition than that for the first-order (for which C^1 for $f(\cdot)$ is enough);
- 2) The derivation of the second-order necessary condition is probably more complicated than that of the first-order situation;
- 3) Usually, in order to establish the second-order necessary condition, one needs to assume that the first-order condition degenerates in some sense.

Very similar phenomenons happen when one establishes the optimality conditions for optimal control problems, though generally it turns out to be much more difficult than that for the above minimization problem.

For the moment, let us return to the deterministic optimal control problem, i.e., the functions $\sigma(\cdot) \equiv 0$, $b(\cdot)$, $f(\cdot)$, $h(\cdot)$, $x(\cdot)$ and $u(\cdot)$ in (1.1)–(1.2) are independent of the sample point ω . Let $\psi(\cdot)$ be the solution to the following ordinary differential equation,

$$(1.9) \quad \begin{cases} \dot{\psi}(t) = -b_x(t, \bar{x}(t), \bar{u}(t))^\top \psi(t) + f_x(t, \bar{x}(t), \bar{u}(t)), & t \in [0, T], \\ \psi(T) = -h_x(\bar{x}(T)). \end{cases}$$

Define the Hamiltonian

$$H(t, x, u, \psi) := \langle \psi, b(t, x, u) \rangle - f(t, x, u), \quad \forall (t, x, u, \psi) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n.$$

Then the following Pontryagin maximum principle ([23]) holds

$$(1.10) \quad H(t, \bar{x}(t), \bar{u}(t), \psi(t)) = \max_{v \in U} H(t, \bar{x}(t), v, \psi(t)), \quad a.e. \ t \in [0, T].$$

The maximum condition (1.10) is a first-order necessary condition for optimal controls. Suppose that, for a.e. $t \in [0, T]$ the maximization problem (1.10) admits a unique solution and the optimal control $\bar{u}(\cdot)$ can be represented as a function $\Upsilon(\cdot, \cdot, \cdot)$ of t , $\bar{x}(t)$ and $\psi(t)$, i.e., $\bar{u}(t) = \Upsilon(t, \bar{x}(t), \psi(t))$ satisfies

$$(1.11) \quad H(t, \bar{x}(t), \Upsilon(t, \bar{x}(t), \psi(t)), \psi(t)) = \max_{v \in U} H(t, \bar{x}(t), v, \psi(t)), \quad a.e. \ t \in [0, T].$$

Then, substituting Υ into the control system (1.1) (with $\sigma \equiv 0$) and the adjoint equation (1.9), we obtain the following two-point boundary-value problem:

$$(1.12) \quad \begin{cases} \dot{\bar{x}}(t) = H_\psi(t, \bar{x}(t), \Upsilon(t, \bar{x}(t), \psi(t)), \psi(t)), & t \in [0, T], \\ \dot{\psi}(t) = -H_x(t, \bar{x}(t), \Upsilon(t, \bar{x}(t), \psi(t)), \psi(t)), & t \in [0, T], \\ \bar{x}(0) = x_0, \ \psi(T) = -h_x(\bar{x}(T)). \end{cases}$$

If both the original optimal control problem and the two-point boundary-value problem (1.12) admit unique solutions, then $\bar{u}(\cdot) = \Upsilon(\cdot, \bar{x}(\cdot), \psi(\cdot))$ is the solution to the original optimal control problem (1.3) where $(\bar{x}(\cdot), \psi(\cdot))$ is the solution to (1.12). It is easy to see that, the uniqueness of the solution to the maximization problem (1.10) (in the first-order necessary condition) plays an important role to reformulate the original optimal control problem into the two-point boundary-value problem (1.12). When this maximization problem admits multi-solutions, the first-order necessary condition is not enough to determine the optimal controls. Indeed, in this cases, the solution map for the maximization problem becomes a set-valued map. When substituting this set-valued map into the control system and the adjoint equation, one obtains a differential inclusion problem, which is usually very hard to solve. When this happens, as in the classical calculus, it is quite useful to analyze further the second-order (or even higher-order) necessary conditions for optimal controls. In the case of deterministic control problems (even in finite dimensions), there are many works devoted to this topic (See [4, 9, 10, 12, 14, 17, 21, 25] and the rich references therein), especially one can find several interesting monographs ([1, 6, 11, 15]) in this direction.

Naturally, one expects to establish the stochastic maximum principle for the optimal control problem (1.3). We refer to [2, 3, 13, 16] and references cited therein for some early works in this respect. Since in this case the Itô integral appears in the control system (1.1), things became much complicated. First, quite different from the equation (1.9), the adjoint equations in the stochastic cases (called backward stochastic differential equations, BSDEs for short) have two unknowns. Second, when the control region is nonconvex, the needle variation, which is essential a perturbation of the optimal control on a measurable set with small measure, is used to derive the optimality conditions. When the diffusion term σ contains the control variable u , the state increment is an infinitesimal of order $1/2$ with respect to ε ($\varepsilon \rightarrow 0^+$) (when the optimal control is perturbed with respect to the time variable on a measurable set with Lebesgue measure ε). Therefore, to obtain the first-order necessary condition

for optimal controls for the general case, the cost functional needs to be expanded up to the second order, and two variational equations and two adjoint equations need to be introduced (See [22]). More precisely, define the Hamiltonian \mathcal{H} by

$$(1.13) \quad \begin{aligned} \mathcal{H}(\omega, t, x, u, y_1, z_1) &= \langle y_1, b(\omega, t, x, u) \rangle + \langle z_1, \sigma(\omega, t, x, u) \rangle - f(\omega, t, x, u), \\ &\forall (\omega, t, x, u, y_1, z_1) \in \Omega \times [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n. \end{aligned}$$

Let $(p_1(\cdot), q_1(\cdot))$ and $(p_2(\cdot), q_2(\cdot))$ be respectively solutions to the following first- and second-order adjoint equations,

$$(1.14) \quad \begin{cases} dp_1(t) = -[b_x(t)^\top p_1(t) + \sigma_x(t)^\top q_1(t) - f_x(t)]dt + q_1(t)dW(t), & t \in [0, T], \\ p_1(T) = -h_x(\bar{x}(T)) \end{cases}$$

and

$$(1.15) \quad \begin{cases} dp_2(t) = -[b_x(t)^\top p_2(t) + p_2(t)b_x(t) + \sigma_x(t)^\top p_2(t)\sigma_x(t) + \sigma_x(t)^\top q_2(t) \\ \quad + q_2(t)\sigma_x(t) + \mathcal{H}_{xx}(t)]dt + q_2(t)dW(t), & t \in [0, T], \\ p_2(T) = -h_{xx}(\bar{x}(T)), \end{cases}$$

where $b_x(t) = b_x(t, \bar{x}(t), \bar{u}(t))$, $\sigma_x(t) = \sigma_x(t, \bar{x}(t), \bar{u}(t))$, $f_x(t) = f_x(t, \bar{x}(t), \bar{u}(t))$, $\mathcal{H}_{xx}(t) = \mathcal{H}_{xx}(t, \bar{x}(t), \bar{u}(t), p_1(t), q_1(t))$. The following first-order necessary condition for the optimal pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is established in [22]:

$$(1.16) \quad \mathbb{H}(t, \bar{x}(t), v) \leq 0, \quad \forall v \in U, \text{ a.e. } (\omega, t) \in \Omega \times [0, T].$$

where

$$\begin{aligned} \mathbb{H}(\omega, t, x, u) &= \mathcal{H}(\omega, t, x, u, p_1(t), q_1(t)) - \mathcal{H}(\omega, t, x, \bar{u}(t), p_1(t), q_1(t)) \\ &\quad + \frac{1}{2} \langle p_2(t)(\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))), \sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t)) \rangle, \\ &\quad (\omega, t, x, u) \in \Omega \times [0, T] \times \mathbb{R}^n \times U. \end{aligned}$$

Similar to the above, if the optimal control $\bar{u}(\cdot)$ can be represented as a function $\Psi(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$ of $(\omega, t, \bar{x}, p_1, q_1, p_2)$ using the condition (1.16) (i.e., $\bar{u}(\omega, t) = \Psi(\omega, t, \bar{x}(t), p_1(t), q_1(t), p_2(t))$) (Note that q_2 does not appear explicitly in the definition of \mathbb{H}), then the optimal control problem can be closely related to the following fully coupled forward backward stochastic differential equation (FBSDE, in short):

$$(1.17) \quad \begin{cases} d\bar{x}(t) = \hat{b}(t)dt + \hat{\sigma}(t)dW(t), & t \in [0, T], \\ dp_1(t) = -[\hat{b}_x(t)^\top p_1(t) + \hat{\sigma}_x(t)^\top q_1(t) - \hat{f}_x(t)]dt + q_1(t)dW(t), & t \in [0, T], \\ dp_2(t) = -[\hat{b}_x(t)^\top p_2(t) + p_2(t)\hat{b}_x(t) + \hat{\sigma}_x(t)^\top p_2(t)\hat{\sigma}_x(t) + \hat{\sigma}_x(t)^\top q_2(t) \\ \quad + q_2(t)\hat{\sigma}_x(t) + \hat{\mathcal{H}}_{xx}(t)]dt + q_2(t)dW(t), & t \in [0, T], \\ x(0) = x_0, \quad p_1(T) = -h_x(\bar{x}(T)), \quad p_2(T) = -h_{xx}(\bar{x}(T)). \end{cases}$$

where $\hat{b}(t) = b(t, \bar{x}(t), \Psi(t, \bar{x}(t), p_1(t), q_1(t), p_2(t)))$, $\hat{\sigma}(t) = \sigma(t, \bar{x}(t), \Psi(t, \bar{x}(t), p_1(t), q_1(t), p_2(t)))$, $\hat{\mathcal{H}}_{xx}(t) = \mathcal{H}_{xx}(t, \bar{x}(t), \Psi(t, \bar{x}(t), p_1(t), q_1(t), p_2(t)))$, similar for $\hat{b}_x(t)$, $\hat{\sigma}_x(t)$, and $\hat{f}_x(t)$. For some more discussions about FBSDEs, we refer to [18].

However, exactly as the deterministic case, the first-order necessary condition is not always effectively to find the stochastic optimal controls. In the preceding discussion, the uniqueness of the solution to (1.16) plays an important role to reduce the original optimal control problem to the FBSDE (1.17). When the problem (1.16) admits multi-solutions, one needs to establish suitable second-order necessary condition for optimal controls as an effective supplement to the first-order condition. As we mentioned before, there exist many works addressing to the corresponding deterministic problems. However, in the stochastic setting, there are only two articles [5] and [24] available before our work [28]. When the diffusion terms do not contain the control variable, Tang [24] derived a pointwise second-order maximum principle for stochastic optimal controls, for which the control regions are allowed to be nonconvex. When the diffusion terms contain the control variable, Bonnans and Silva [5] established some integral-type (rather than pointwise) second-order necessary conditions for stochastic optimal controls with convex control constraints. In [28], we found that, quite different from the deterministic setting, there exist some essential difficulties in deriving the *pointwise* second-order necessary condition from an integral-type one whenever the diffusion terms contain the control variable, even for the special case of convex control constraints, and obtained a positive result for this case under some assumptions in terms of the Malliavin calculus.

The main purpose of this paper is to establish some pointwise second-order necessary conditions for stochastic optimal controls in the general cases, i.e., the control regions are allowed to be nonconvex and both the drift and diffusion terms contain the control variable. Stimulated by [22], it is easy to see that, in order to obtain the second-order optimality condition for the general case, one needs to expand the cost functional up to the forth order, and introduce four variational equations and four adjoint equations. This is the main difference between the present paper and the previous related works (i.e., [5, 24, 28]). On the other hand, the solutions of the variational equations appear in the second-order terms (in the sense of the perturbation measure) of the variational formulation with respect to the optimal controls, and it seems to us that, they cannot be eliminated by introducing new adjoint equations. When the diffusion terms of the control systems contain the control variables, similar to the convex control constraint cases, the Lebesgue differentiation theorem cannot be used directly to derive the pointwise second-order necessary condition from the variational formulation (See [28, Subsection 3.2] for a detailed explanation). This is another difference between this paper and [24] addressing to the case of the diffusion term independent of the control variable. In this paper, first we establish a variational formulation of (1.3) with respect to the optimal controls. Then, using this variational formulation and the martingale representation theorem, we derive a second-order necessary condition for stochastic optimal controls. Further, under some conditions, we refine this result and obtain a pointwise second-order necessary condition. Note that the analysis in this paper is much complicated than that in [28] though some of the ideas and techniques are the same in these two papers.

The rest of this paper is organized as follows. In Section 2, we collect some notation and concepts. In Section 3, we introduce the related variational equations and adjoint equations. In Section 4, we state the main results of this paper and present some remarks and examples. Section 5 is devoted to proving our main results. Finally, the proofs of two technical results are given in Appendixes A and B, respectively.

Partial results in this paper have been announced in [27] without proofs.

2. Preliminaries. Let $m, n, d, h, l \in \mathbb{N}$. Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively the inner product and norm in \mathbb{R}^n or \mathbb{R}^m , which can be identified from the contexts. For any $\alpha, \beta \in [1, +\infty)$, denote by $L_{\mathcal{F}_T}^\beta(\Omega; \mathbb{R}^n)$ the space of \mathcal{F}_T -measurable random variables ξ such that $\mathbb{E} |\xi|^\beta < +\infty$, by $L^\beta(\Omega \times [0, T]; \mathbb{R}^n)$ the space of $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable processes φ such that $\|\varphi\|_\beta := [\mathbb{E} \int_0^T |\varphi(t)|^\beta dt]^\frac{1}{\beta} < +\infty$, by $L_{\mathbb{F}}^\beta(\Omega; L^\alpha(0, T; \mathbb{R}^n))$ the space of $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable, \mathbb{F} -adapted processes φ such that $\|\varphi\|_{\alpha, \beta} := [\mathbb{E} (\int_0^T |\varphi(t)|^\alpha dt)^\frac{\beta}{\alpha}]^\frac{1}{\beta} < +\infty$, by $L_{\mathbb{F}}^\beta(\Omega; C([0, T]; \mathbb{R}^n))$ the space of $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable, \mathbb{F} -adapted continuous processes φ such that $\|\varphi\|_{\infty, \beta} := [\mathbb{E} (\sup_{t \in [0, T]} |\varphi(t)|^\beta)]^\frac{1}{\beta} < +\infty$, by $L^\infty(\Omega \times [0, T]; \mathbb{R}^n)$ the space of $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable processes φ such that $\|\varphi\|_\infty := \text{ess sup}_{(\omega, t) \in \Omega \times [0, T]} |\varphi(\omega, t)| < +\infty$, and by $L^\beta(0, T; L_{\mathbb{F}}^\beta(\Omega \times [0, T]; \mathbb{R}^n))$ the $\mathcal{F} \otimes \mathcal{B}([0, T] \times [0, T])$ measurable maps φ such that for any $t \in [0, T]$, $\varphi(\cdot, t)$ is \mathbb{F} -adapted and $\|\varphi\|_\beta := [\mathbb{E} \int_0^T \int_0^T |\varphi(s, t)|^\beta ds dt]^\frac{1}{\beta} < +\infty$.

Let $\mathbb{D}^{1,2}(\mathbb{R}^n) \subset L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n)$ be the space of Malliavin differentiable random variables, and for any $\xi \in \mathbb{D}^{1,2}(\mathbb{R}^n)$ denote by $\mathcal{D}\xi$ its Malliavin derivative. Denote by $\mathbb{L}^{1,2}(\mathbb{R}^n)$ the subspace of $L^2(\Omega \times [0, T]; \mathbb{R}^n)$ whose elements satisfy the following conditions.

- (i) For almost every $t \in [0, T]$, $\varphi(t, \cdot) \in \mathbb{D}^{1,2}(\mathbb{R}^n)$,
 - (ii) $(\omega, t, s) \rightarrow D_s \varphi(t, \omega)$ admits an $\mathcal{F} \otimes \mathcal{B}([0, T] \times [0, T])$ -measurable version, and
 - (iii) $\|\varphi\|_{1,2} := [\mathbb{E} \int_0^T |\varphi(t)|^2 dt + \mathbb{E} \int_0^T \int_0^T |D_s \varphi(t)|^2 ds dt]^\frac{1}{2} < +\infty$,
- where $\mathcal{D}\varphi(t, \cdot)$ is the Malliavin derivative of the random variable $\varphi(t, \cdot)$. Denote by $\mathbb{L}_2^{1,2}(\mathbb{R}^n)$ the subspace of $L^2(\Omega \times [0, T]; \mathbb{R}^n)$ whose elements are Malliavin differentiable almost everywhere and their Malliavin derivatives have suitable continuity. More precisely, write

$$\begin{aligned} \mathbb{L}_{2+}^{1,2}(\mathbb{R}^n) &:= \left\{ \varphi(\cdot) \in \mathbb{L}^{1,2}(\mathbb{R}^n) \mid \exists \nabla^+ \varphi(\cdot) \in L^2(\Omega \times [0, T]; \mathbb{R}^n) \text{ such that} \right. \\ &\quad \left. f_\varepsilon(s) := \sup_{s < t < (s+\varepsilon) \wedge T} \mathbb{E} |D_s \varphi(t) - \nabla^+ \varphi(s)|^2 < +\infty, \text{ a.e. } s \in [0, T], \right. \\ &\quad \left. f_\varepsilon(\cdot) \text{ is measurable on } [0, T] \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \rightarrow 0^+} \int_0^T f_\varepsilon(s) ds = 0 \right\}, \\ \mathbb{L}_{2-}^{1,2}(\mathbb{R}^n) &:= \left\{ \varphi(\cdot) \in \mathbb{L}^{1,2}(\mathbb{R}^n) \mid \exists \nabla^- \varphi(\cdot) \in L^2(\Omega \times [0, T]; \mathbb{R}^n) \text{ such that} \right. \\ &\quad \left. g_\varepsilon(s) := \sup_{(s-\varepsilon) \vee 0 < t < s} \mathbb{E} |D_s \varphi(t) - \nabla^- \varphi(s)|^2 < +\infty, \text{ a.e. } s \in [0, T], \right. \\ &\quad \left. g_\varepsilon(\cdot) \text{ is measurable on } [0, T] \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \rightarrow 0^+} \int_0^T g_\varepsilon(s) ds = 0 \right\}. \end{aligned}$$

Denote $\mathbb{L}_2^{1,2}(\mathbb{R}^n) = \mathbb{L}_{2+}^{1,2}(\mathbb{R}^n) \cap \mathbb{L}_{2-}^{1,2}(\mathbb{R}^n)$. For any $\varphi(\cdot) \in \mathbb{L}_2^{1,2}(\mathbb{R}^n)$, denote $\nabla \varphi(\cdot) = \nabla^+ \varphi(\cdot) + \nabla^- \varphi(\cdot)$. When φ is \mathbb{F} -adapted, $D_s \varphi(t) = 0$ for any $t < s$. In this case, $\nabla^- \varphi(\cdot) = 0$, and $\nabla \varphi(\cdot) = \nabla^+ \varphi(\cdot)$. Denote by $\mathbb{L}_{2, \mathbb{F}}^{1,2}(\mathbb{R}^n)$ the set of all \mathbb{F} -adapted processes in $\mathbb{L}_2^{1,2}(\mathbb{R}^n)$. We refer to [20] for more materials on this topic.

Denote by $L(\Pi_d \mathbb{R}^n; \mathbb{R}^m)$ the d -linear maps from $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_d$ to \mathbb{R}^m . Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n , $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ be the standard basis of \mathbb{R}^m . Any Λ in $L(\Pi_d \mathbb{R}^n; \mathbb{R}^m)$ is uniquely determined by the numbers

$$\lambda^{ij_1 \dots j_d} := \langle \Lambda(e_{j_1}, \dots, e_{j_d}), \mathbf{e}_i \rangle, \quad i = 1, \dots, m, \quad j_k = 1, \dots, n, \quad k = 1, \dots, d.$$

We define the norm of Λ by

$$|\Lambda| := \left[\sum_{i,j,k,k=1,\dots,d} \left(\lambda^{ij_1,\dots,j_d} \right)^2 \right]^{\frac{1}{2}}.$$

Let $\Gamma \in L(\Pi_h \mathbb{R}^n; \mathbb{R}^n)$ and $\Theta \in L(\Pi_l \mathbb{R}^n; \mathbb{R}^n)$. We denote by $\Lambda \circ_i \Gamma$ the composition of Λ with Γ at the i th position ($i = 1, \dots, d$), i.e.,

$$\begin{aligned} \Lambda \circ_i \Gamma(x_1, \dots, x_{d+h-1}) &= \Lambda(x_1, \dots, \Gamma_i(x_i, \dots, x_{i+h-1}), \dots, x_{d+h-1}), \\ x_k &\in \mathbb{R}^n, \quad k = 1, \dots, d+h-1, \end{aligned}$$

and we denote by $\Lambda \circ_{i,j}(\Gamma, \Theta)$ the composition of Λ with Γ and Θ at the i th and the j th positions ($i \neq j$, $i, j = 1, \dots, d$), i.e.,

$$\begin{aligned} \Lambda \circ_{i,j}(\Gamma, \Theta)(x_1, \dots, x_{d+h+l-2}) \\ = \Lambda(x_1, \dots, \Gamma_i(x_i, \dots, x_{i+h-1}), \dots, \Theta_j(x_{j+h-1}, \dots, x_{j+h+l-2}), \dots, x_{d+h+l-2}), \\ x_k \in \mathbb{R}^n, \quad k = 1, \dots, d+h+l-2. \end{aligned}$$

In a similar way, if $y, z \in \mathbb{R}^n$, we denote

$$\begin{aligned} \Lambda \bullet_i y(x_1, \dots, x_{d-1}) &= \Lambda(x_1, \dots, y_i, \dots, x_{d-1}), \quad x_k \in \mathbb{R}^n, \quad k = 1, \dots, d-1, \\ \Lambda \bullet_{i,j}(y, z)(x_1, \dots, x_{d-2}) &= \Lambda(x_1, \dots, y_i, \dots, z_j, \dots, x_{d-2}), \quad x_k \in \mathbb{R}^n, \quad k = 1, \dots, d-2. \end{aligned}$$

Denote by $L_{\mathbb{F}}^2(\Omega; L^1(0, T; L(\Pi_d \mathbb{R}^n; \mathbb{R})))$ the space of $\mathcal{F} \otimes \mathcal{B}([0, T])/\mathcal{B}(L(\Pi_d \mathbb{R}^n; \mathbb{R}))$ -measurable processes φ such that $\|\varphi\|_{1,2} := [\mathbb{E} (\int_0^T |\varphi(t)|^2 dt)]^{\frac{1}{2}} < +\infty$ and by $L_{\mathbb{F}}^2(\Omega; L^2(0, T; L(\Pi_d \mathbb{R}^n; \mathbb{R})))$ the space of $\mathcal{F} \otimes \mathcal{B}([0, T])/\mathcal{B}(L(\Pi_d \mathbb{R}^n; \mathbb{R}))$ -measurable processes φ such that $\|\varphi\|_2 := [\mathbb{E} \int_0^T |\varphi(t)|^2 dt]^{\frac{1}{2}} < +\infty$. We give below an Itô formula for multi-linear function-valued stochastic processes, which is an easy extension of the classical Itô formula (Hence we omit its proof).

LEMMA 2.1. *Let $P(\cdot)$ be an $L(\Pi_d \mathbb{R}^n; \mathbb{R})$ -valued process of the form*

$$P(t) = P_0 + \int_0^t A(s) ds + \int_0^t B(s) dW(s), \quad t \in [0, T]$$

where $A(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^1(0, T; L(\Pi_d \mathbb{R}^n; \mathbb{R})))$, $B(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^2(0, T; L(\Pi_d \mathbb{R}^n; \mathbb{R})))$, and let $x(\cdot)$ be an \mathbb{R}^n -valued process such that

$$x(t) = x_0 + \int_0^t f(s) ds + \int_0^t g(s) dW(s), \quad t \in [0, T],$$

where $f(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^1(0, T; \mathbb{R}^n))$, $g(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^2(0, T; \mathbb{R}^n))$. Then the following Itô formula holds.

$$P(T)(x(T), \dots, x(T)) - P_0(x_0, \dots, x_0)$$

$$\begin{aligned}
&= \int_0^T \left[\sum_{i=1}^d P(t)(x(t), \dots, f_i(t), \dots, x(t)) + A(t)(x(t), \dots, x(t)) \right. \\
&\quad + \sum_{i=1}^d B(t)(x(t), \dots, g_i(t), \dots, x(t)) \\
&\quad \left. + \sum_{i=1}^d \sum_{j=i+1}^d P(t)(x(t), \dots, g_i(t), \dots, g_j(t), \dots, x(t)) \right] dt \\
(2.1) \quad &+ \int_0^T \left[B(t)(x(t), \dots, x(t)) + \sum_{i=1}^d P(t)(x(t), \dots, g_i(t), \dots, x(t)) \right] dW(t).
\end{aligned}$$

3. Variational formulations. In this section, we establish a second-order (with respect to the perturbation measure) Taylor expansion of the cost function at the optimal control $\bar{u}(\cdot)$. Firstly, we recall some known estimates for stochastic differential equations.

LEMMA 3.1. ([19, Proposition 2.1]) *Suppose that there exists a constant $L > 0$ such that for $\varphi = b, \sigma$ and any $x, \tilde{x} \in \mathbb{R}^n, u \in U$,*

$$(3.1) \quad \begin{cases} |\varphi(t, x, u) - \varphi(t, \tilde{x}, u)| \leq L|x - \tilde{x}|, & \text{a.s. a.e. } t \in [0, T], \\ |\varphi(t, 0, u)| \leq L, & \text{a.s. a.e. } t \in [0, T]. \end{cases}$$

Then for any $\beta \geq 1$, $u(\cdot) \in \mathcal{U}_{ad}$ and initial datum $x_0 \in \mathbb{R}^n$, the state equation (1.1) admits a unique solution $x(\cdot) \in L_{\mathbb{F}}^{\beta}(\Omega; C([0, T]; \mathbb{R}^n))$, and for some constant $C = C(\beta, L, T) > 0$ the following estimate holds:

$$\begin{aligned}
&\mathbb{E} \left[\sup_{s \in [0, t]} |x(s)|^{\beta} \right] \\
(3.2) \quad &\leq C \mathbb{E} \left[|x_0|^{\beta} + \left(\int_0^t |b(s, 0, u(s))| ds \right)^{\beta} + \left(\int_0^t |\sigma(s, 0, u(s))|^2 ds \right)^{\frac{\beta}{2}} \right].
\end{aligned}$$

Further, if $\bar{x}(\cdot)$ is the unique solution corresponding to $(\bar{x}_0, \bar{u}(\cdot)) \in \mathbb{R}^n \times \mathcal{U}_{ad}$, then

$$\begin{aligned}
&\mathbb{E} \left[\sup_{s \in [0, t]} |x(s) - \bar{x}(s)|^{\beta} \right] \\
&\leq C \mathbb{E} \left[|x_0 - \bar{x}_0|^{\beta} + \left(\int_0^t |b(s, \bar{x}(s), u(s)) - b(s, \bar{x}(s), \bar{u}(s))| ds \right)^{\beta} \right. \\
(3.3) \quad &\quad \left. + \left(\int_0^t |\sigma(s, \bar{x}(s), u(s)) - \sigma(s, \bar{x}(s), \bar{u}(s))|^2 ds \right)^{\frac{\beta}{2}} \right].
\end{aligned}$$

In what follows, we assume that

(C1) The control region $U \subset \mathbb{R}^m$ is nonempty and bounded.

(C2) Functions b, σ, f , and h satisfy

- (i) For any $(x, u) \in \mathbb{R}^n \times U$, the stochastic processes $b(\cdot, x, u) : [0, T] \times \Omega \rightarrow \mathbb{R}^n$, $\sigma(\cdot, x, u) : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ and $f(\cdot, x, u) : \Omega \times [0, T] \rightarrow \mathbb{R}$ are $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable and \mathbb{F} -adapted. $h(x, \cdot) : \Omega \rightarrow \mathbb{R}$ is \mathcal{F}_T -measurable.

- (ii) For almost all $(\omega, t) \in \Omega \times [0, T]$ and any $u \in U$, the map $x \mapsto (b(\omega, t, x, u), \sigma(\omega, t, x, u), f(\omega, t, x, u))$ is continuously differentiable up to the forth order, and there exist a constant $L > 0$ and a modulus of continuity $\tilde{\omega} : [0, \infty) \rightarrow [0, \infty)$ such that for a.e. $(\omega, t) \in \Omega \times [0, T]$, and all $x, \tilde{x} \in \mathbb{R}^n, u, \tilde{u} \in U, \varphi = b, \sigma, f$,

$$\begin{cases} |\varphi(t, 0, 0)| \leq L, & |\varphi(t, x, u) - \varphi(t, \tilde{x}, \tilde{u})| \leq L|x - \tilde{x}| + \tilde{\omega}(|u - \tilde{u}|), \\ |\varphi_x(t, x, u) - \varphi_x(t, \tilde{x}, \tilde{u})| \leq L|x - \tilde{x}| + \tilde{\omega}(|u - \tilde{u}|), \\ |\varphi_{xx}(t, x, u) - \varphi_{xx}(t, \tilde{x}, \tilde{u})| \leq L|x - \tilde{x}| + \tilde{\omega}(|u - \tilde{u}|), \\ |\varphi_{xxx}(t, x, u) - \varphi_{xxx}(t, \tilde{x}, \tilde{u})| \leq L|x - \tilde{x}| + \tilde{\omega}(|u - \tilde{u}|), \\ |\varphi_{xxxx}(t, x, u) - \varphi_{xxxx}(t, \tilde{x}, \tilde{u})| \leq L|x - \tilde{x}| + \tilde{\omega}(|u - \tilde{u}|). \end{cases}$$

- (iii) $h(\cdot)$ is continuously differentiable up to the forth order (a.s.), and there exists a constant $L > 0$ such that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |h(x)| &\leq L(1 + |x|^4), & |h_x(x)| &\leq L(1 + |x|^3), \\ |h_{xx}(x)| &\leq L(1 + |x|^2), & |h_{xxx}(x)| &\leq L(1 + |x|), & |h_{xxxx}(x)| &\leq L, \quad a.s. \end{aligned}$$

Obviously, for $\varphi = b, \sigma, f$, when (C1)–(C2) are satisfied, for a.e. $(\omega, t) \in \Omega \times [0, T]$,

$$|\varphi_x(t, x, u)| + |\varphi_{xx}(t, x, u)| + |\varphi_{xxx}(t, x, u)| + |\varphi_{xxxx}(t, x, u)| \leq L,$$

for all $(x, u) \in \mathbb{R}^n \times U$, and the controlled stochastic differential equation (1.1) admits a unique solution for any $u(\cdot) \in \mathcal{U}_{ad}$ and the cost functional is well-defined.

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair, $u(\cdot) \in \mathcal{U}_{ad}$ be an admissible control, $E_\varepsilon \subset [0, T]$ be a measurable set with measure $|E_\varepsilon| = \varepsilon$ for a given $\varepsilon \in (0, T)$. Define

$$u^\varepsilon(t) = \begin{cases} u(t), & t \in E_\varepsilon, \\ \bar{u}(t), & t \in [0, T] \setminus E_\varepsilon. \end{cases}$$

Let $x^\varepsilon(\cdot)$ be the state with respect to the control $u^\varepsilon(\cdot)$ and let $\delta x(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot)$. For $\varphi = b, \sigma, f$, write $\varphi_x(t) = \varphi_x(t, \bar{x}(t), \bar{u}(t))$, $\varphi_{xx}(t) = \varphi_{xx}(t, \bar{x}(t), \bar{u}(t))$, $\varphi_{xxx}(t) = \varphi_{xxx}(t, \bar{x}(t), \bar{u}(t))$, $\varphi_{xxxx}(t) = \varphi_{xxxx}(t, \bar{x}(t), \bar{u}(t))$, and put

$$\begin{aligned} \delta\varphi(t) &= \varphi(t, \bar{x}(t), u(t)) - \varphi(t, \bar{x}(t), \bar{u}(t)), \\ \delta\varphi_x(t) &= \varphi_x(t, \bar{x}(t), u(t)) - \varphi_x(t, \bar{x}(t), \bar{u}(t)), \\ \delta\varphi_{xx}(t) &= \varphi_{xx}(t, \bar{x}(t), u(t)) - \varphi_{xx}(t, \bar{x}(t), \bar{u}(t)), \\ \delta\varphi_{xxx}(t) &= \varphi_{xxx}(t, \bar{x}(t), u(t)) - \varphi_{xxx}(t, \bar{x}(t), \bar{u}(t)). \end{aligned}$$

Now, we introduce the following four variational equations:

$$(3.4) \quad \begin{cases} dy_1^\varepsilon(t) = b_x(t)y_1^\varepsilon(t)dt + [\sigma_x(t)y_1^\varepsilon(t) + \delta\sigma(t)\chi_{E_\varepsilon}(t)]dW(t), & t \in [0, T], \\ y_1^\varepsilon(0) = 0; \end{cases}$$

$$(3.5) \quad \begin{cases} dy_2^\varepsilon(t) = \left[b_{xx}(t)y_2^\varepsilon(t) + \frac{1}{2}b_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta b(t)\chi_{E_\varepsilon}(t) \right] dt \\ \quad + \left[\sigma_x(t)y_2^\varepsilon(t) + \frac{1}{2}\sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) \right. \\ \quad \left. + \delta\sigma_x(t)y_1^\varepsilon(t)\chi_{E_\varepsilon}(t) \right] dW(t), & t \in [0, T], \\ y_2^\varepsilon(0) = 0; \end{cases}$$

$$(3.6) \quad \left\{ \begin{aligned} dy_3^\varepsilon(t) &= \left\{ b_x(t)y_3^\varepsilon(t) + \frac{1}{2}[b_{xx}(t)(y_1^\varepsilon(t) + y_2^\varepsilon(t), y_2^\varepsilon(t)) \right. \\ &\quad + b_{xx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t))] + \frac{1}{6}b_{xxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + \delta b_x(t)y_1^\varepsilon(t)\chi_{E_\varepsilon}(t) \Big\} dt \\ &\quad + \left\{ \sigma_x(t)y_3^\varepsilon(t) + \frac{1}{2}[\sigma_{xx}(t)(y_1^\varepsilon(t) + y_2^\varepsilon(t), y_2^\varepsilon(t)) \right. \\ &\quad + \sigma_{xx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t))] + \frac{1}{6}\sigma_{xxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + \delta\sigma_x(t)y_2^\varepsilon(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta\sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t))\chi_{E_\varepsilon}(t) \Big\} dW(t), \\ &\quad t \in [0, T], \\ y_3^\varepsilon(0) &= 0; \end{aligned} \right.$$

$$(3.7) \quad \left\{ \begin{aligned} dy_4^\varepsilon(t) &= \left\{ b_x(t)y_4^\varepsilon(t) + \frac{1}{2}[b_{xx}(t)(y_1^\varepsilon(t) + y_2^\varepsilon(t) + y_3^\varepsilon(t), y_3^\varepsilon(t)) \right. \\ &\quad + b_{xx}(t)(y_3^\varepsilon(t), y_1^\varepsilon(t) + y_2^\varepsilon(t))] + \frac{1}{6}[b_{xxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_2^\varepsilon(t)) \\ &\quad + b_{xxx}(t)(y_1^\varepsilon(t), y_2^\varepsilon(t), y_1^\varepsilon(t)) + b_{xxx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + b_{xxx}(t)(y_1^\varepsilon(t), y_2^\varepsilon(t), y_2^\varepsilon(t)) + b_{xxx}(t)(y_2^\varepsilon(t), y_2^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + b_{xxx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t), y_2^\varepsilon(t)) + b_{xxx}(t)(y_2^\varepsilon(t), y_2^\varepsilon(t), y_2^\varepsilon(t))] \\ &\quad + \frac{1}{24}b_{xxxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + \delta b_x(t)y_2^\varepsilon(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta b_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t))\chi_{E_\varepsilon}(t) \Big\} dt \\ &\quad + \left\{ \sigma_x(t)y_4^\varepsilon(t) + \frac{1}{2}[\sigma_{xx}(t)(y_1^\varepsilon(t) + y_2^\varepsilon(t) + y_3^\varepsilon(t), y_3^\varepsilon(t)) \right. \\ &\quad + \sigma_{xx}(t)(y_3^\varepsilon(t), y_1^\varepsilon(t) + y_2^\varepsilon(t))] + \frac{1}{6}[\sigma_{xxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_2^\varepsilon(t)) \\ &\quad + \sigma_{xxx}(t)(y_1^\varepsilon(t), y_2^\varepsilon(t), y_1^\varepsilon(t)) + \sigma_{xxx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + \sigma_{xxx}(t)(y_1^\varepsilon(t), y_2^\varepsilon(t), y_2^\varepsilon(t)) + \sigma_{xxx}(t)(y_2^\varepsilon(t), y_2^\varepsilon(t), y_1^\varepsilon(t)) \\ &\quad + \sigma_{xxx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t), y_2^\varepsilon(t)) + \sigma_{xxx}(t)(y_2^\varepsilon(t), y_2^\varepsilon(t), y_2^\varepsilon(t))] \\ &\quad + \frac{1}{24}\sigma_{xxxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta\sigma_x(t)y_3^\varepsilon(t)\chi_{E_\varepsilon}(t) \\ &\quad + \frac{1}{2}[\delta\sigma_{xx}(t)(y_1^\varepsilon(t) + y_2^\varepsilon(t), y_2^\varepsilon(t)) + \delta\sigma_{xx}(t)(y_2^\varepsilon(t), y_1^\varepsilon(t))] \chi_{E_\varepsilon}(t) \\ &\quad + \frac{1}{6}\delta\sigma_{xxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t))\chi_{E_\varepsilon}(t) \Big\} dW(t), \quad t \in [0, T], \\ y_4^\varepsilon(0) &= 0. \end{aligned} \right.$$

Denote

$$\begin{aligned} \xi(\cdot) &:= y_1^\varepsilon(\cdot) + y_2^\varepsilon(\cdot) + y_3^\varepsilon(\cdot) + y_4^\varepsilon(\cdot), & \eta(\cdot) &:= y_1^\varepsilon(\cdot) + y_2^\varepsilon(\cdot) + y_3^\varepsilon(\cdot), \\ \gamma(\cdot) &:= y_1^\varepsilon(\cdot) + y_2^\varepsilon(\cdot), & r_1(\cdot) &:= \delta x(\cdot) - y_1^\varepsilon(\cdot), \\ r_2(\cdot) &:= \delta x(\cdot) - \gamma(\cdot), & r_3(\cdot) &:= \delta x(\cdot) - \eta(\cdot), \\ r_4(\cdot) &:= \delta x(\cdot) - \xi(\cdot). \end{aligned}$$

From (3.4)–(3.7) and Lemma 3.1, we obtain the following result.

LEMMA 3.2. *Let (C1) and (C2) hold. Then, for any $\beta \geq 1$, $\varepsilon \in (0, T)$, $\varepsilon \rightarrow 0^+$, the following estimates hold:*

$$\begin{aligned} \|y_1^\varepsilon\|_{\infty, \beta}^\beta &\leq C\varepsilon^{\frac{\beta}{2}}, & \|r_1\|_{\infty, \beta}^\beta &\leq C\varepsilon^\beta, \\ \|y_2^\varepsilon\|_{\infty, \beta}^\beta &\leq C\varepsilon^\beta, & \|r_2\|_{\infty, \beta}^\beta &\leq C\varepsilon^{\frac{3\beta}{2}}, \\ \|y_3^\varepsilon\|_{\infty, \beta}^\beta &\leq C\varepsilon^{\frac{3\beta}{2}}, & \|r_3\|_{\infty, \beta}^\beta &\leq C\varepsilon^{2\beta}, \\ \|y_4^\varepsilon\|_{\infty, \beta}^\beta &\leq C\varepsilon^{2\beta}, & \|r_4\|_{\infty, \beta}^\beta &\leq C\varepsilon^{\frac{5\beta}{2}}, \\ \|\delta x\|_{\infty, \beta}^\beta &\leq \varepsilon^{\frac{\beta}{2}}. \end{aligned}$$

Proof. See Appendix A. \square

Further, we obtain the following Taylor expansion for the cost functional with respect to the control perturbation.

LEMMA 3.3. *Let (C1) and (C2) hold. Then,*

$$\begin{aligned} &J(u^\varepsilon) - J(\bar{u}) \\ &= \mathbb{E} \int_0^T \left[f_x(t)\xi(t) + \frac{1}{2}f_{xx}(t)(\eta(t), \eta(t)) + \frac{1}{6}f_{xxx}(t)(\gamma(t), \gamma(t), \gamma(t)) \right. \\ &\quad + \frac{1}{24}f_{xxxx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta f(t)\chi_{E_\varepsilon}(t) \\ &\quad + \delta f_x(t)\gamma(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta f_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t))\chi_{E_\varepsilon}(t) \Big] dt \\ &\quad + \mathbb{E} \left[h_x(\bar{x}(T))\xi(T) + \frac{1}{2}h_{xx}(\bar{x}(T))(\eta(T), \eta(T)) \right. \\ &\quad + \frac{1}{6}h_{xxx}(\bar{x}(T))(\gamma(T), \gamma(T), \gamma(T)) \\ &\quad \left. + \frac{1}{24}h_{xxxx}(\bar{x}(T))(y_1^\varepsilon(T), y_1^\varepsilon(T), y_1^\varepsilon(T), y_1^\varepsilon(T)) \right] + o(\varepsilon^2) \quad (\varepsilon \rightarrow 0^+). \end{aligned} \tag{3.8}$$

Proof. Similar to the proof of Lemma 3.2, we only consider the 1-dimensional case. By Taylor's formulation,

$$\begin{aligned} &f(t, x^\varepsilon(t), u^\varepsilon(t)) - f(t, \bar{x}(t), \bar{u}(t)) \\ &= f(t, x^\varepsilon(t), \bar{u}(t)) - f(t, \bar{x}(t), \bar{u}(t)) + f(t, \bar{x}(t), u^\varepsilon(t)) - f(t, \bar{x}(t), \bar{u}(t)) \\ &\quad + f(t, x^\varepsilon(t), u^\varepsilon(t)) - f(t, \bar{x}(t), u^\varepsilon(t)) - f(t, x^\varepsilon(t), \bar{u}(t)) + f(t, \bar{x}(t), \bar{u}(t)) \\ &= f_x(t)\delta x(t) + \frac{1}{2}f_{xx}(t)\delta x(t)^2 + \frac{1}{6}f_{xxx}(t)\delta x(t)^3 \\ &\quad + \frac{1}{6} \int_0^1 \theta^3 f_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta + \delta f(t)\chi_{E_\varepsilon}(t) \\ &\quad + f_x(t, \bar{x}(t), u^\varepsilon(t))\delta x(t) + \frac{1}{2}f_{xx}(t, \bar{x}(t), u^\varepsilon(t))\delta x(t)^2 \\ &\quad + \frac{1}{2} \int_0^1 \theta^2 f_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \delta x(t)^3 d\theta - f_x(t)\delta x(t) \\ &\quad - \frac{1}{2}f_{xx}(t)\delta x(t)^2 - \frac{1}{2} \int_0^1 \theta^2 f_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \\ &= f_x(t)\delta x(t) + \frac{1}{2}f_{xx}(t)\delta x(t)^2 + \frac{1}{6}f_{xxx}(t)\delta x(t)^3 \\ &\quad + \frac{1}{6} \int_0^1 \theta^3 f_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \end{aligned}$$

$$\begin{aligned}
& +\delta f(t)\chi_{E_\varepsilon}(t) + \delta f_x(t)\delta x(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta f_{xx}(t)\delta x(t)^2\chi_{E_\varepsilon}(t) \\
& + \frac{1}{2}\int_0^1 \theta^2 \left(f_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
& \quad \left. - f_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta,
\end{aligned}$$

and

$$\begin{aligned}
& h(x^\varepsilon(T)) - h(\bar{x}(T)) \\
& = h_x(\bar{x}(T))\delta x(T) + \frac{1}{2}h_{xx}(\bar{x}(T))\delta x(T)^2 + \frac{1}{6}h_{xxx}(\bar{x}(T))\delta x(T)^3 \\
& \quad + \frac{1}{6}\int_0^1 \theta^3 h_{xxxx}(\theta\bar{x}(T) + (1-\theta)x^\varepsilon(T))\delta x(T)^4 d\theta.
\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}
& J(u^\varepsilon) - J(\bar{u}) \\
& = \mathbb{E} \int_0^T \left[f_x(t)\xi(t) + \frac{1}{2}f_{xx}(t)\eta(t)^2 + \frac{1}{6}f_{xxx}(t)\gamma(t)^3 + \frac{1}{24}f_{xxxx}(t)y_1^\varepsilon(t)^4 \right. \\
& \quad \left. + \delta f(t)\chi_{E_\varepsilon}(t) + \delta f_x(t)\gamma(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta f_{xx}(t)y_1^\varepsilon(t)^2\chi_{E_\varepsilon}(t) \right] dt \\
& + \mathbb{E} \left[h_x(\bar{x}(T))\xi(T) + \frac{1}{2}h_{xx}(\bar{x}(T))\eta(T)^2 \right. \\
& \quad \left. + \frac{1}{6}h_{xxx}(\bar{x}(T))\gamma(T)^3 + \frac{1}{24}h_{xxxx}(\bar{x}(T))y_1^\varepsilon(T)^4 \right] \\
& + \mathbb{E} \int_0^T \left[f_x(t)r_4(t) + \frac{1}{2}f_{xx}(t)(\delta x(t)^2 - \eta(t)^2) + \frac{1}{6}f_{xxx}(t)(\delta x(t)^3 - \gamma(t)^3) \right. \\
& \quad \left. + \frac{1}{6}\int_0^1 \theta^3 f_{xxxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta - \frac{1}{24}f_{xxxx}(t)y_1^\varepsilon(t)^4 \right. \\
& \quad \left. + \delta f_x(t)r_2(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta f_{xx}(t)(\delta x(t)^2 - y_1^\varepsilon(t)^2)\chi_{E_\varepsilon}(t) \right. \\
& \quad \left. + \frac{1}{2}\int_0^1 \theta^2 \left(f_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \\
& \quad \quad \left. \left. - f_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta \right] dt \\
& + \mathbb{E} \left[h_x(\bar{x}(T))r_4(T) + \frac{1}{2}h_{xx}(\bar{x}(T))(\delta x(T)^2 - \eta(T)^2) \right. \\
& \quad \left. + \frac{1}{6}h_{xxx}(\bar{x}(T))(\delta x(T)^3 - \gamma(T)^3) \right. \\
& \quad \left. + \frac{1}{6}\int_0^1 \theta^3 h_{xxxx}(\theta\bar{x}(T) + (1-\theta)x^\varepsilon(T))\delta x(T)^4 d\theta - \frac{1}{24}h_{xxxx}(\bar{x}(T))y_1^\varepsilon(T)^4 \right] \\
& = \mathbb{E} \int_0^T \left[f_x(t)\xi(t) + \frac{1}{2}f_{xx}(t)\eta(t)^2 + \frac{1}{6}f_{xxx}(t)\gamma(t)^3 + \frac{1}{24}f_{xxxx}(t)y_1^\varepsilon(t)^4 \right. \\
& \quad \left. + \delta f(t)\chi_{E_\varepsilon}(t) + \delta f_x(t)\gamma(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta f_{xx}(t)y_1^\varepsilon(t)^2\chi_{E_\varepsilon}(t) \right] dt \\
& + \mathbb{E} \left[h_x(\bar{x}(T))\xi(T) + \frac{1}{2}h_{xx}(\bar{x}(T))\eta(T)^2 \right]
\end{aligned}$$

$$+\frac{1}{6}h_{xxx}(\bar{x}(T))\gamma(T)^3 + \frac{1}{24}h_{xxxx}(\bar{x}(T))y_1^\varepsilon(T)^4] + o(\varepsilon^2) \quad (\varepsilon \rightarrow 0^+).$$

This completes the proof of Lemma 3.3. \square

To establish the variational formulation for the optimal control $\bar{u}(\cdot)$, in addition to the adjoint equations (1.14)–(1.15), the following two adjoint equations are also needed:

$$(3.9) \quad \left\{ \begin{array}{l} dp_3(t) = - \left[\sum_{k=1}^3 p_3(t) \circ_k b_x(t) + \sum_{k=1}^2 \sum_{l=k+1}^3 p_3(t) \circ_{k,l} (\sigma_x(t), \sigma_x(t)) \right. \\ \quad + \sum_{k=1}^3 q_3(t) \circ_k \sigma_x(t) + \frac{3}{2} \sum_{k=1}^2 \left(p_2(t) \circ_k b_{xx}(t) + q_2(t) \circ_k \sigma_{xx}(t) \right) \\ \quad + \frac{3}{2} \left(p_2(t) \circ_{1,2} (\sigma_x(t), \sigma_{xx}(t)) + p_2(t) \circ_{1,2} (\sigma_{xx}(t), \sigma_x(t)) \right) \\ \quad \left. + \mathcal{H}_{xxx}(t) \right] dt + q_3(t) dW(t), \quad t \in [0, T], \\ p_3(T) = -h_{xxx}(\bar{x}(T)); \end{array} \right.$$

and

$$(3.10) \quad \left\{ \begin{array}{l} dp_4(t) = - \left[\sum_{k=1}^4 p_4(t) \circ_k b_x(t) + \sum_{k=1}^3 \sum_{l=k+1}^4 p_4(t) \circ_{k,l} (\sigma_x(t), \sigma_x(t)) \right. \\ \quad + \sum_{k=1}^4 q_4(t) \circ_k \sigma_x(t) + 2 \sum_{k=1}^3 \left(p_3(t) \circ_k b_{xx}(t) + q_3(t) \circ_k \sigma_{xx}(t) \right) \\ \quad + 2 \sum_{k=1}^3 \sum_{l=1, l \neq k}^3 p_3(t) \circ_{k,l} (\sigma_x(t), \sigma_{xx}(t)) + 2 \sum_{k=1}^2 p_2(t) \circ_k b_{xxx}(t) \\ \quad + 2 \left(p_2(t) \circ_{1,2} (\sigma_x(t), \sigma_{xxx}(t)) + p_2(t) \circ_{1,2} (\sigma_{xxx}(t), \sigma_x(t)) \right) \\ \quad + 3 p_2(t) \circ_{1,2} (\sigma_{xx}(t), \sigma_{xx}(t)) + 2 \sum_{k=1}^2 q_2(t) \circ_k \sigma_{xxx}(t) \\ \quad \left. + \mathcal{H}_{xxxx}(t) \right] dt + q_4(t) dW(t), \quad t \in [0, T], \\ p_4(T) = -h_{xxxx}(\bar{x}(T)), \end{array} \right.$$

where the Hamiltonian \mathcal{H} is defined by (1.13), and

$$\begin{aligned} \mathcal{H}_{xxx}(t) &= \mathcal{H}_{xxx}(\omega, t, \bar{x}(t), \bar{u}(t), p_1(t), q_1(t)), \\ \mathcal{H}_{xxxx}(t) &= \mathcal{H}_{xxxx}(\omega, t, \bar{x}(t), \bar{u}(t), p_1(t), q_1(t)). \end{aligned}$$

By the existence and regularity results for BSDEs (see [7]), for any $\beta \geq 1$, the adjoint equations (3.9)–(3.10) admit unique solutions, respectively, and

$$\begin{aligned} (p_3(\cdot), q_3(\cdot)) &\in L_{\mathbb{F}}^{\beta}(\Omega; C([0, T]; L(\Pi_3 \mathbb{R}^n; \mathbb{R}))) \times L_{\mathbb{F}}^{\beta}(\Omega; L^2(0, T; L(\Pi_3 \mathbb{R}^n; \mathbb{R}))), \\ (p_4(\cdot), q_4(\cdot)) &\in L_{\mathbb{F}}^{\beta}(\Omega; C([0, T]; L(\Pi_4 \mathbb{R}^n; \mathbb{R}))) \times L_{\mathbb{F}}^{\beta}(\Omega; L^2(0, T; L(\Pi_4 \mathbb{R}^n; \mathbb{R}))). \end{aligned}$$

Using the Taylor expansion of the cost functional established in Lemma 3.3 and the duality relationship between the variational equations (3.4)–(3.7) and the adjoint equations (1.14)–(1.15) and (3.9)–(3.10), we obtain a variational formulation for the cost functional. In order to short the expression of this formulation, we introduce some more notations.

Let the Hamiltonian be defined by (1.13). Write

$$\begin{aligned} \mathbb{S}(\omega, t, x, u, y_2, z_2) &= \frac{1}{2} \sum_{k=1}^2 \left[y_2 \bullet_k b(\omega, t, x, u) + z_2 \bullet_k \sigma(\omega, t, x, u) \right], \\ (\omega, t, x, u, y_2, z_2) &\in \Omega \times [0, T] \times \mathbb{R}^n \times U \times L(\Pi_2 \mathbb{R}^n; \mathbb{R}) \times L(\Pi_2 \mathbb{R}^n; \mathbb{R}); \\ \mathbb{T}(\omega, t, x, u, y_3, z_3) &= \frac{1}{3} \sum_{k=1}^3 \left[y_3 \bullet_k b(\omega, t, x, u) + z_3 \bullet_k \sigma(\omega, t, x, u) \right], \\ (\omega, t, x, u, y_3, z_3) &\in \Omega \times [0, T] \times \mathbb{R}^n \times U \times L(\Pi_3 \mathbb{R}^n; \mathbb{R}) \times L(\Pi_3 \mathbb{R}^n; \mathbb{R}), \end{aligned}$$

and denote

$$\begin{aligned} &\mathbb{S}(\omega, t, x, u) \\ &= \mathbb{H}_x(\omega, t, x, u) + \mathbb{S}(\omega, t, x, u, p_2(t), q_2(t)) - \mathbb{S}(\omega, t, x, \bar{u}(t), p_2(t), q_2(t)) \\ &\quad + \frac{1}{2} [p_2(t) \circ \sigma_x(\omega, t, x, \bar{u}(t))] \bullet_2 (\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))) \\ &\quad + \frac{1}{2} [p_2(t) \circ \sigma_x(\omega, t, x, \bar{u}(t))] \bullet_1 (\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))) \\ &\quad + \frac{1}{6} \sum_{k=1}^2 \sum_{l=k+1}^3 p_3(t) \bullet_{k,l} (\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t)), \sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))), \\ &\mathbb{T}(\omega, t, x, u) \\ &= \mathbb{S}_x(\omega, t, x, u) + \mathbb{S}_x(\omega, t, x, u, p_2(t), q_2(t)) - \mathbb{S}_x(\omega, t, x, \bar{u}(t), p_2(t), q_2(t)) \\ &\quad + \mathbb{T}(\omega, t, x, u, p_3(t), q_3(t)) - \mathbb{T}(\omega, t, x, \bar{u}(t), p_3(t), q_3(t)) \\ &\quad + \frac{1}{2} \sum_{k=1}^2 \sum_{l=1, l \neq k}^2 p_2(t) \circ_{k,l} (\sigma_x(\omega, t, x, \bar{u}(t)), \sigma_x(\omega, t, x, u) - \sigma_x(\omega, t, x, \bar{u}(t))) \\ &\quad + \frac{1}{6} \sum_{k=1}^3 \sum_{l=1, l \neq k}^3 [p_3(t) \circ_k (\sigma_x(\omega, t, x, u) - \sigma_x(\omega, t, x, \bar{u}(t)))] \\ &\quad \bullet_l (\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))) \\ &\quad + \frac{1}{3} \sum_{k=1}^3 \sum_{l=1, l \neq k}^3 [p_3(t) \circ_k \sigma_x(\omega, t, x, \bar{u}(t))] \bullet_l (\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))) \\ &\quad + \frac{1}{12} \sum_{k=1}^3 \sum_{l=k+1}^4 p_4(t) \bullet_{k,l} (\sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t)), \\ &\quad \sigma(\omega, t, x, u) - \sigma(\omega, t, x, \bar{u}(t))), \\ &\quad (\omega, t, x, u) \in \Omega \times [0, T] \times \mathbb{R}^n \times U, \end{aligned}$$

where, $(p_1(\cdot), q_1(\cdot))$ and $(p_2(\cdot), q_2(\cdot))$ are respectively the solutions to (1.14) and (1.15), $(p_3(\cdot), q_3(\cdot))$ and $(p_4(\cdot), q_4(\cdot))$ are respectively the solutions to (3.9) and (3.10).

We have the following variational formulation for the cost functional.

PROPOSITION 3.4. *Let (C1) and (C2) hold. Then,*

$$\begin{aligned}
 & J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) \\
 &= -\mathbb{E} \int_0^T \left[\mathbb{H}(t, \bar{x}(t), u(t)) + \langle \mathbb{S}(t, \bar{x}(t), u(t)), \gamma(t) \rangle \right. \\
 (3.11) \quad & \left. + \frac{1}{2} \langle \mathbb{T}(t, \bar{x}(t), u(t)) y_1^\varepsilon(t), y_1^\varepsilon(t) \rangle \right] \chi_{E_\varepsilon}(t) dt + o(\varepsilon^2), \quad (\varepsilon \rightarrow 0^+).
 \end{aligned}$$

Proof. See Appendix B. \square

4. Second-order necessary conditions. In this section, we establish some second-order necessary conditions for stochastic singular optimal controls in the sense of Pontryagin-type maximum principle. Firstly, we introduce the concept of the singular control (The corresponding concept for deterministic control systems can be found in [10] and the references cited therein).

DEFINITION 4.1. *An admissible control $\tilde{u}(\cdot)$ is called a singular control in the sense of Pontryagin-type maximum principle on a control region V , if V is a nonempty subset of U and*

$$\begin{aligned}
 (4.1) \quad 0 &= \mathbb{H}(t, \tilde{x}(t), v) \\
 &= \mathcal{H}(t, \tilde{x}(t), v, \tilde{p}_1(t), \tilde{q}_1(t)) - \mathcal{H}(t, \tilde{x}(t), \tilde{u}(t), \tilde{p}_1(t), \tilde{q}_1(t)) \\
 &\quad + \frac{1}{2} \langle \tilde{p}_2(t)(\sigma(t, \tilde{x}(t), v) - \sigma(t, \tilde{x}(t), \tilde{u}(t))), \sigma(t, \tilde{x}(t), v) - \sigma(t, \tilde{x}(t), \tilde{u}(t)) \rangle, \\
 &\quad \forall v \in V, \text{ a.s., a.e. } t \in [0, T].
 \end{aligned}$$

where $\tilde{x}(\cdot)$ is the state with respect to $\tilde{u}(\cdot)$, and $(\tilde{p}_1(\cdot), \tilde{q}_1(\cdot))$, $(\tilde{p}_2(\cdot), \tilde{q}_2(\cdot))$ are the adjoint processes given respectively by (1.14) and (1.15) with $(\bar{x}(\cdot), \bar{u}(\cdot))$ replaced by $(\tilde{x}(\cdot), \tilde{u}(\cdot))$. If the singular control $\tilde{u}(\cdot)$ is also optimal, we call it a singular optimal control.

In the sequel, we shall fix the control subset $V \subset U$ appeared in Definition 4.1.

Remark 4.1. In [28], we introduced the concept of singular control in the classical sense. Let us recall that, an admissible control $\tilde{u}(\cdot)$ is called a singular control in the classical sense if $\tilde{u}(\cdot)$ satisfies

$$(4.2) \quad \begin{cases} \mathcal{H}_u(t, \tilde{x}(t), \tilde{u}(t), \tilde{p}_1(t), \tilde{q}_1(t)) = 0, & \text{a.s., a.e. } t \in [0, T], \\ \mathcal{H}_{uu}(t, \tilde{x}(t), \tilde{u}(t), \tilde{p}_1(t), \tilde{q}_1(t)) + \sigma_u(t, \tilde{x}(t), \tilde{u}(t))^\top \tilde{p}_2(t) \sigma_u(t, \tilde{x}(t), \tilde{u}(t)) = 0, \\ & \text{a.s., a.e. } t \in [0, T]. \end{cases}$$

If $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is an optimal pair, the first-order necessary condition (1.16) says that the map

$$v \mapsto \mathbb{H}(\omega, t, \tilde{x}(t), v), \quad v \in U$$

admits its maximum at $\tilde{u}(t)$ for a.e. $(\omega, t) \in \Omega \times [0, T]$. A singular control in the classical sense is the one that satisfies trivially the first- and second-order necessary conditions (for a.e. $(\omega, t) \in \Omega \times [0, T]$) for the maximization problem

$$\max_{v \in U} \mathbb{H}(t, \tilde{x}(t), v).$$

Obviously, when the set V is open and $\tilde{u}(t) \in V$, a.e. $(\omega, t) \in \Omega \times [0, T]$, any singular control in the sense of Pontryagin-type maximum principle satisfies (4.2), that is, \tilde{u} is also a singular control in the classical sense, but not vice versa.

Remark 4.2. Since in this paper we consider the case of diffusion term containing the control variable, in (4.1) there exists the second order term

$$\frac{1}{2} \langle \tilde{p}_2(t)(\sigma(t, \tilde{x}(t), v) - \sigma(t, \tilde{x}(t), \tilde{u}(t))), \sigma(t, \tilde{x}(t), v) - \sigma(t, \tilde{x}(t), \tilde{u}(t)) \rangle.$$

When the diffusion term independent of the control variable this term is equal to 0. In this case, Definition 4.1 reduces to Definition 2.1. in [24].

We need the following simple result.

LEMMA 4.2. Let (C1) and (C2) hold. Then $\mathbb{S}(\cdot, \bar{x}(\cdot), u(\cdot)) \in L_{\mathbb{F}}^4(\Omega; L^2([0, T]; \mathbb{R}^n))$ and $\mathbb{T}(\cdot, \bar{x}(\cdot), u(\cdot)) \in L_{\mathbb{F}}^2(\Omega; L^2([0, T]; L(\Pi \mathbb{R}^n; \mathbb{R})))$ for any $u(\cdot) \in \mathcal{U}_{ad}$.

Proof. It is sufficient to prove that

$$\mathbb{E} \left[\int_0^T |\mathbb{S}(t, \bar{x}(t), u(t))|^2 dt \right]^2 < \infty$$

and

$$\mathbb{E} \int_0^T |\mathbb{T}(t, \bar{x}(t), u(t))|^2 dt < \infty.$$

By (C1)–(C2), there exists a constant C such that, for $\varphi = b, \delta, f$,

$$|\varphi_x(t)| \leq C, \quad |\delta\varphi(t)| \leq C, \quad \text{and} \quad |\delta\varphi_x(t)| \leq C, \quad \text{a.e. } (\omega, t) \in \Omega \times [0, T].$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[\int_0^T |\mathbb{S}(t, \bar{x}(t), u(t))|^2 dt \right]^2 \\ &= \mathbb{E} \left[\int_0^T |\delta b_x(t) \bullet_1 p_1(t) + \delta\sigma_x(t) \bullet_1 q_1(t) - \delta f_x(t) \right. \\ & \quad + \frac{1}{2} \sum_{k=1}^2 p_2(t) \bullet_k \delta b(t) + \frac{1}{2} \sum_{k=1}^2 q_2(t) \bullet_k \delta\sigma(t) + \frac{1}{2} [p_2(t) \circ_1 \sigma_x(t)] \bullet_2 \delta\sigma(t) \\ & \quad + \frac{1}{2} [p_2(t) \circ_2 \sigma_x(t)] \bullet_1 \delta\sigma(t) + \frac{1}{2} [p_2(t) \circ_1 \delta\sigma_x(t)] \bullet_2 \delta\sigma(t) \\ & \quad \left. + \frac{1}{2} [p_2(t) \circ_2 \delta\sigma_x(t)] \bullet_1 \delta\sigma(t) + \frac{1}{6} \sum_{k=1}^2 \sum_{l=k+1}^3 p_3(t) \bullet_{k,l} (\delta\sigma(t), \delta\sigma(t)) \right|^2 dt \Big]^2 \\ &\leq C + C \mathbb{E} \left[\int_0^T (|p_1(t)|^2 + |q_1(t)|^2 + |p_2(t)|^2 + |q_2(t)|^2 + |p_3(t)|^2) dt \right]^2 \\ &\leq C + C (\|p_1\|_{\infty,4}^4 + \|q_1\|_{2,4}^4 + \|p_2\|_{\infty,4}^4 + \|q_2\|_{2,4}^4 + \|p_3\|_{\infty,4}^4) \\ &< \infty. \end{aligned}$$

In a similar way, we can prove that

$$\mathbb{E} \int_0^T |\mathbb{T}(t, \bar{x}(t), u(t))|^2 dt < \infty.$$

□

By Lemma 4.2, it follows that $\mathbb{S}(\cdot, \bar{x}(\cdot), v) \in L^2_{\mathbb{F}}(\Omega; L^2([0, T]; \mathbb{R}^n))$ for any $v \in V \subset U$. By [28, Lemma 3.8], there exists a $\phi_v \in L^2(0, T; L^2_{\mathbb{F}}(\Omega \times [0, T]; \mathbb{R}^n))$ such that

$$(4.3) \quad \mathbb{S}(t, \bar{x}(t), v) = \mathbb{E} \mathbb{S}(t, \bar{x}(t), v) + \int_0^t \phi_v(s, t) dW(s), \quad \text{a.s., a.e. } t \in [0, T].$$

Denote by $\Phi(\cdot)$ the solution to the following stochastic differential equation

$$(4.4) \quad \begin{cases} d\Phi(t) = b_x(t)\Phi(t)dt + \sigma_x(t)\Phi(t)dW(t), & t \in [0, T], \\ \Phi(0) = I, \end{cases}$$

where I is the identity matrix in $\mathbb{R}^{n \times n}$.

Using the martingale representation formula (4.3), we obtain the following second-order necessary condition:

THEOREM 4.3. *Let (C1) and (C2) hold. If $\bar{u}(\cdot)$ is a singular optimal control in the sense of Pontryagin-type maximum principle on the control subset $V \subset U$, then, for any $v \in V$, it holds that*

$$(4.5) \quad \begin{aligned} & \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), b(\tau, \bar{x}(\tau), v) - b(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \\ & + \partial_{\tau}^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right) \\ & + \frac{1}{2} \mathbb{E} \left\langle \mathbb{T}(\tau, \bar{x}(\tau), v) (\sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \right. \\ & \left. \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \leq 0, \quad \text{a.e. } \tau \in [0, T]. \end{aligned}$$

where

$$(4.6) \quad \begin{aligned} & \frac{1}{2} \partial_{\tau}^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right) \\ & := \limsup_{\theta \rightarrow 0^+} \frac{1}{\theta^2} \mathbb{E} \int_{\tau}^{\tau+\theta} \int_{\tau}^t \left\langle \phi_v(s, t), \right. \\ & \quad \left. \Phi(\tau)\Phi(s)^{-1} (\sigma(s, \bar{x}(s), v) - \sigma(s, \bar{x}(s), \bar{u}(s))) \right\rangle ds dt, \end{aligned}$$

$\phi_v(\cdot, \cdot)$ is determined by (4.3).

The proof of Theorem 4.3 will be given in Subsection 5.1.

Note that the second-order necessary condition (4.5) is only a pointwise type condition with respect to the time variable $t \in [0, T]$. To obtain the pointwise second-order necessary conditions with respect to both the time t and the sample point $\omega \in \Omega$, similar to the first part of our work (see [28]), we need the following regularity condition.

(C3) For any $v \in V$, $\mathbb{S}(\cdot, \bar{x}(\cdot), v) \in L^{1,2}_{2,\mathbb{F}}(\mathbb{R}^n)$, and the map $v \mapsto \nabla \mathbb{S}(t, \bar{x}(t), v)$ is continuous on V a.s., a.e. $t \in [0, T]$.

We have the following result.

THEOREM 4.4. *Let (C1)–(C3) hold. If $\bar{u}(\cdot)$ is a singular optimal control in the sense of Pontryagin-type maximum principle on the control subset $V \subset U$, then, for*

a.e. $\tau \in [0, T]$, it holds that

$$\begin{aligned}
 & \langle \mathbb{S}(\tau, \bar{x}(\tau), v), b(\tau, \bar{x}(\tau), v) - b(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \\
 & + \langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \\
 & + \frac{1}{2} \langle \mathbb{T}(\tau, \bar{x}(\tau), v) (\sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \\
 (4.7) \quad & \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \leq 0, \quad \forall v \in V, \text{ a.s.}
 \end{aligned}$$

The proof of Theorem 4.4 will be given in Subsection 5.2.

As an easy consequence of Theorem 4.4, the following pointwise second-order condition immediately holds.

COROLLARY 4.5. *Let (C1)–(C2) hold. If $\bar{u}(\cdot)$ is a singular optimal control in the sense of Pontryagin-type maximum principle on the control subset $V \subset U$ and*

$$(4.8) \quad \mathbb{S}(t, \bar{x}(t), v) = 0, \quad \forall v \in V, \text{ a.s., a.e. } t \in [0, T],$$

then, for a.e. $\tau \in [0, T]$, it holds that

$$\begin{aligned}
 (4.9) \quad & \langle \mathbb{T}(\tau, \bar{x}(\tau), v) (\sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \\
 & \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \leq 0, \quad \forall v \in V, \text{ a.s.}
 \end{aligned}$$

Remark 4.3. *When the diffusion term is independent of the control variable, $\sigma(t, \bar{x}(t), v) - \sigma(t, \bar{x}(t), \bar{u}(t)) = 0$ for any $(\omega, t) \in \Omega \times [0, T]$. Therefore,*

$$\langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \equiv 0,$$

$$\langle \mathbb{T}(\tau, \bar{x}(\tau), v) (\sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \equiv 0,$$

and the condition (4.7) is reduced to

$$\langle \mathbb{S}(\tau, \bar{x}(\tau), v), b(\tau, \bar{x}(\tau), v) - b(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \leq 0, \quad \forall v \in V, \text{ a.s., a.e. } \tau \in [0, T],$$

where, in this case,

$$\begin{aligned}
 \mathbb{S}(\omega, t, x, u) &= \mathcal{H}_x(\omega, t, x, u, p_1(t), q_1(t)) - \mathcal{H}_x(\omega, t, x, \bar{u}(t), p_1(t), q_1(t)), \\
 &+ \frac{1}{2} p_2(t) (b(\omega, t, x, u) - b(\omega, t, x, \bar{u}(t))) \\
 &+ \frac{1}{2} (b(\omega, t, x, u) - b(\omega, t, x, \bar{u}(t)))^\top p_2(t), \\
 &(\omega, t, x, u) \in \Omega \times [0, T] \times \mathbb{R}^n \times U.
 \end{aligned}$$

The corresponding result coincides with [24, Theorem 2.1]. In addition, since the diffusion term is independent of the control variable, $y_1^\varepsilon(t) \equiv 0$, and hence

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \mathbb{S}(t, \bar{x}(t), v) y_1^\varepsilon(t) \chi_{E_\varepsilon}(t) dt = 0.$$

In this case, it is unnecessary to introduce the regularity assumption (C3) to prove the desired condition (4.7).

Remark 4.4. In Theorem 4.4, we obtain a pointwise second-order necessary condition for stochastic optimal controls under relatively weak assumptions on the control

set U through the perturbation technique of needle variation. However, this approach needs considerably high smoothness assumptions on the coefficients b , σ , f , and h with respect to the state variable x (differentiable with respect x up to the forth order). Furthermore, four adjoint equations are introduced to represent this condition. When the set U has good structure such that the first- and second-order adjacent sets of U on the boundary point of U is nonempty (but U is still allowed to be nonconvex), some perturbation technique from the classical variational analysis can be used to establish the second-order necessary conditions for stochastic optimal controls under lower regularity assumption on the coefficients b , σ , f , and h (with respect to the state variable x) and only two adjoint equations are introduced to derive the second-order necessary conditions. We refer the reader to [8] for a detailed discussion in this respect.

Two illustrative examples are as follows.

Example 4.1. Let

$$\begin{cases} dx(t) = b(x(t))u(t)dt + u(t)dW(t), \\ x(0) = 0, \end{cases} \quad t \in [0, 1],$$

$U = \{-1, 0, 1\}$, and let

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E} \int_0^1 |u(t)|^2 dt - \frac{1}{2}\mathbb{E} |x(1)|^2.$$

Assume that $b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is bounded and continuously differentiable up to order 5 with bounded derivatives, $b_x(0) > 0$. Then, the conditions (C1)–(C2) hold.

For the above optimal control problem, the Hamiltonian is defined by

$$\mathcal{H}(t, x, u, p_1, q_1) = p_1 b(x)u + q_1 u - \frac{1}{2}u^2,$$

$(t, x, u, p_1, q_1) \in [0, 1] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}$.

Let $(\bar{x}(t), \bar{u}(t)) \equiv (0, 0)$. The four adjoint equations with respect to $(\bar{x}(\cdot), \bar{u}(\cdot))$ are given below:

$$\begin{cases} dp_1(t) = q_1(t)dW(t), & t \in [0, 1], \\ p_1(1) = 0; \end{cases} \quad \begin{cases} dp_2(t) = q_2(t)dW(t), & t \in [0, 1], \\ p_2(1) = 1; \end{cases}$$

$$\begin{cases} dp_3(t) = q_3(t)dW(t), & t \in [0, 1], \\ p_3(1) = 0; \end{cases} \quad \begin{cases} dp_4(t) = q_4(t)dW(t), & t \in [0, 1], \\ p_4(1) = 0. \end{cases}$$

It is easy to check that

$$\begin{aligned} (p_1(t), q_1(t)) &= (0, 0), & (p_2(t), q_2(t)) &= (1, 0), \\ (p_3(t), q_3(t)) &= (0, 0), & (p_4(t), q_4(t)) &= (0, 0), \quad \forall (\omega, t) \in \Omega \times [0, 1]; \end{aligned}$$

and,

$$\begin{aligned} \mathbb{H}(t, \bar{x}(t), v) &= 0, & \mathbb{S}(t, \bar{x}(t), v) &= b(0)v, & \mathbb{T}(t, \bar{x}(t), v) &= 2b_x(0)v, \\ & & & & \forall v \in U, & \forall (\omega, t) \in \Omega \times [0, 1]. \end{aligned}$$

Thus, $\bar{u}(t) \equiv 0$ is a singular control in the sense of Pontryagin-type maximum principle on U .

Let $v = 1$, we have

$$\mathbb{S}(t, \bar{x}(t), v) = b(0) = \mathbb{E} \mathbb{S}(t, \bar{x}(t), v).$$

In this case, $\nabla \mathbb{S}(t, \bar{x}(t), v) \equiv 0$, and

$$\begin{aligned} & \langle \mathbb{S}(\tau, \bar{x}(\tau), v), b(\tau, \bar{x}(\tau), v) - b(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \\ & + \langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \\ & + \frac{1}{2} \langle \mathbb{T}(\tau, \bar{x}(\tau), v) (\sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \\ & \quad \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \rangle \\ & = b(0)^2 + b_x(0) \\ & > 0, \quad \forall (\omega, t) \in \Omega \times [0, 1]. \end{aligned}$$

Therefore, by Theorem 4.4, $\bar{u}(t) \equiv 0$ is not an optimal control.

Example 4.2. Let

$$\begin{cases} dx(t) = (u(t) - 1)dt + (x(t) - u(t))dW(t), & t \in [0, 1], \\ x(0) = 1, \end{cases}$$

$U = \{-1, 0, 1\}$, and let

$$J(u(\cdot)) = \frac{1}{24} \mathbb{E} |x(1) - 1|^4.$$

Obviously, $(\bar{x}(\cdot), \bar{u}(\cdot)) \equiv (1, 1)$ is the optimal pair. The four adjoint equations with respect to $(\bar{x}(\cdot), \bar{u}(\cdot))$ are as follows:

$$\begin{cases} dp_1(t) = -q_1(t)dt + q_1(t)dW(t), & t \in [0, 1], \\ p_1(1) = 0; \end{cases}$$

$$\begin{cases} dp_2(t) = -[p_2(t) + 2q_2(t)]dt + q_2(t)dW(t), & t \in [0, 1], \\ p_2(1) = 0; \end{cases}$$

$$\begin{cases} dp_3(t) = -[3p_3(t) + 3q_3(t)]dt + q_3(t)dW(t), & t \in [0, 1], \\ p_3(1) = 0; \end{cases}$$

and

$$\begin{cases} dp_4(t) = -[6p_4(t) + 4q_4(t)]dt + q_4(t)dW(t), & t \in [0, 1], \\ p_4(1) = -1. \end{cases}$$

An easy computation shows that

$$\begin{aligned} (p_1(t), q_1(t)) &= (0, 0), \quad (p_2(t), q_2(t)) = (0, 0), \\ (p_3(t), q_3(t)) &= (0, 0), \quad (p_4(t), q_4(t)) = (-e^{6-6t}, 0), \quad \forall (\omega, t) \in \Omega \times [0, 1]. \end{aligned}$$

Then, we have

$$\begin{aligned} \mathbb{H}(t, \bar{x}(t), v) &= 0, \quad \mathbb{S}(t, \bar{x}(t), v) = 0, \quad \mathbb{T}(t, \bar{x}(t), v) = -\frac{1}{2}e^{6-6t}(v-1)^2, \\ &\quad \forall v \in U, \quad \forall (\omega, t) \in \Omega \times [0, 1], \end{aligned}$$

and

$$\begin{aligned} & \left\langle \mathbb{T}(t, \bar{x}(t), v) (\sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \right. \\ & \quad \left. \sigma(\tau, \bar{x}(\tau), v) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \\ &= -\frac{1}{2} e^{6-6t} (v-1)^4 \leq 0, \quad \forall v \in U, \quad \forall (\omega, t) \in \Omega \times [0, 1]. \end{aligned}$$

Therefore, $\bar{u}(t) \equiv 1$ is a singular optimal control on U , and the second-order necessary condition (4.9) holds.

5. Proofs of the main results. This section is devoted to proving the main results of this paper, i.e., Theorems 4.3 and 4.4. We need a known result.

LEMMA 5.1. ([28, Lemma 4.1]) *Let $\Phi(\cdot), \Psi(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$. Then, for a.e. $\tau \in [0, T]$, it holds that*

$$(5.1) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \Phi(\tau), \int_{\tau}^t \Psi(s) ds \right\rangle dt = \frac{1}{2} \mathbb{E} \langle \Phi(\tau), \Psi(\tau) \rangle,$$

$$(5.2) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \Phi(t), \int_{\tau}^t \Psi(s) ds \right\rangle dt = \frac{1}{2} \mathbb{E} \langle \Phi(\tau), \Psi(\tau) \rangle.$$

5.1. Proof of Theorem 4.3. Since $u(t) \equiv v$, $v \in U$ is an admissible control, in this subsection, we shall still denote by $\delta\varphi(t)$ the increment $\varphi(t, \bar{x}(t), v) - \varphi(t, \bar{x}(t), \bar{u}(t))$ and by $\delta\varphi_x(t)$ the increment $\varphi_x(t, \bar{x}(t), v) - \varphi_x(t, \bar{x}(t), \bar{u}(t))$ for $\varphi = b, \sigma, f$. We only need to prove the condition (4.5) holds for a.e. $\tau \in [0, T]$. Let $\tau \in [0, T]$, $\varepsilon \in (0, T - \tau)$ and $E_\varepsilon = [\tau, \tau + \varepsilon) \subset [0, T]$. For any fixed $v \in V$, define

$$u^\varepsilon(t) = \begin{cases} v, & t \in E_\varepsilon, \\ \bar{u}(t), & t \in [0, T] \setminus E_\varepsilon. \end{cases}$$

Clearly, $u^\varepsilon(\cdot) \in \mathcal{U}_{ad}$. Since $\bar{u}(\cdot)$ is a singular control on V in the sense of Pontryagin-type maximum principle,

$$\mathbb{H}(t, \bar{x}(t), v) = 0, \quad a.e. (\omega, t) \in \Omega \times [0, T].$$

Then, by Proposition 3.4, we have

$$\begin{aligned} 0 &\geq \frac{J(\bar{u}(\cdot)) - J(u^\varepsilon(\cdot))}{\varepsilon^2} \\ &= \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left[\mathbb{H}(t, \bar{x}(t), v) + \left\langle \mathbb{S}(t, \bar{x}(t), v), \gamma(t) \right\rangle \right. \\ &\quad \left. + \frac{1}{2} \left\langle \mathbb{T}(t, \bar{x}(t), v) y_1^\varepsilon(t), y_1^\varepsilon(t) \right\rangle \right] \chi_{E_\varepsilon}(t) dt + o(1) \quad (\varepsilon \rightarrow 0^+) \\ &= \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left[\left\langle \mathbb{S}(t, \bar{x}(t), v), y_1^\varepsilon(t) + y_2^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) \right. \\ &\quad \left. + \frac{1}{2} \left\langle \mathbb{T}(t, \bar{x}(t), v) y_1^\varepsilon(t), y_1^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) \right] dt + o(1), \quad (\varepsilon \rightarrow 0^+). \end{aligned}$$

Now, we divide the proof of (4.5) into 4 steps.

Step 1: In this step, we prove that

$$\begin{aligned}
 & \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v), y_1^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) dt \\
 (5.3) \quad & = \frac{1}{2} \partial_\tau^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \delta\sigma(\tau) \right), \quad a.e. \ \tau \in [0, T].
 \end{aligned}$$

By [26, Theorem 1.6.14, p. 47)], $y_1^\varepsilon(\cdot)$ has the following explicit representation:

$$\begin{aligned}
 y_1^\varepsilon(t) &= -\Phi(t) \int_0^t \Phi(s)^{-1} \sigma_x(s) \delta\sigma(s) \chi_{E_\varepsilon}(s) ds \\
 (5.4) \quad &+ \Phi(t) \int_0^t \Phi(s)^{-1} \delta\sigma(s) \chi_{E_\varepsilon}(s) dW(s).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 & \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v), y_1^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) dt \\
 &= -\frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_x(s) \delta\sigma(s) ds \right\rangle dt \\
 (5.5) \quad &+ \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_\tau^t \Phi(s)^{-1} \delta\sigma(s) dW(s) \right\rangle dt.
 \end{aligned}$$

By Lemma 5.1, it follows that

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0^+} \left[-\frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_x(s) \delta\sigma(s) ds \right\rangle dt \right] \\
 (5.6) \quad &= -\frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma_x(\tau) \delta\sigma(\tau) \right\rangle, \quad a.e. \ \tau \in [0, T].
 \end{aligned}$$

Next, by (4.4), we deduce that

$$\begin{aligned}
 & \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_\tau^t \Phi(s)^{-1} \delta\sigma(s) dW(s) \right\rangle dt \\
 &= \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(\tau) \int_\tau^t \Phi(s)^{-1} \delta\sigma(s) dW(s) \right\rangle dt \\
 &+ \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \int_\tau^t b_x(s) \Phi(s) ds \cdot \right. \\
 &\quad \left. \int_\tau^t \Phi(s)^{-1} \delta\sigma(s) dW(s) \right\rangle dt \\
 &+ \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \int_\tau^t \sigma_x(s) \Phi(s) dW(s) \cdot \right. \\
 &\quad \left. \int_\tau^t \Phi(s)^{-1} \delta\sigma(s) dW(s) \right\rangle dt. \\
 (5.7)
 \end{aligned}$$

By (4.3) and (4.6), it holds that

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(\tau) \int_\tau^t \Phi(s)^{-1} \delta\sigma(s) dW(s) \right\rangle dt$$

$$\begin{aligned}
&= \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{E} \mathbb{S}(t, \bar{x}(t), v) + \int_0^t \phi_v(s, t) dW(s), \right. \\
&\quad \left. \int_{\tau}^t \Phi(\tau) \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt \\
&= \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \left\langle \phi_v(s, t), \Phi(\tau) \Phi(s)^{-1} \delta \sigma(s) \right\rangle ds dt \\
(5.8) \quad &= \frac{1}{2} \partial_{\tau}^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \delta \sigma(\tau) \right) \quad a.e. \ \tau \in [0, T].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \int_{\tau}^t b_x(s) \Phi(s) ds \cdot \right. \right. \\
&\quad \left. \left. \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt \right| \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \int_{\tau}^{\tau+\varepsilon} \left[\left(\mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 \right)^{\frac{1}{2}} \left(\mathbb{E} \left| \int_{\tau}^t b_x(s) \Phi(s) ds \right|^4 \right)^{\frac{1}{4}} \cdot \right. \\
&\quad \left. \left(\mathbb{E} \left| \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right|^4 \right)^{\frac{1}{4}} \right] dt \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon^2} \int_{\tau}^{\tau+\varepsilon} (t - \tau)^{\frac{3}{2}} \left(\mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 \right)^{\frac{1}{2}} dt \\
(5.9) \quad &= 0 \quad a.e. \ \tau \in [t_i, T].
\end{aligned}$$

Also,

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \int_{\tau}^t \sigma_x(s) \Phi(s) dW(s) \cdot \right. \right. \\
&\quad \left. \left. \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt - \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma_x(\tau) \delta \sigma(\tau) \right\rangle \right| \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v) - \mathbb{S}(\tau, \bar{x}(\tau), v), \right. \right. \\
&\quad \left. \left. \int_{\tau}^t \sigma_x(s) \Phi(s) dW(s) \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt \right| \\
&\quad + \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \int_{\tau}^t \sigma_x(s) \Phi(s) dW(s) \cdot \right. \right. \\
&\quad \left. \left. \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt - \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma_x(\tau) \delta \sigma(\tau) \right\rangle \right| \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \int_{\tau}^{\tau+\varepsilon} \left[\mathbb{E} |\mathbb{S}(t, \bar{x}(t), v) - \mathbb{S}(\tau, \bar{x}(\tau), v)|^2 \right]^{\frac{1}{2}} \cdot \\
&\quad \left[\mathbb{E} \left(\int_{\tau}^t |\sigma_x(s) \Phi(s)|^2 ds \right)^2 \right]^{\frac{1}{4}} \left[\mathbb{E} \left(\int_{\tau}^t |\Phi(s)^{-1} \delta \sigma(s)|^2 ds \right)^2 \right]^{\frac{1}{4}} dt \\
&\quad + \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma_x(s) \delta \sigma(s) - \sigma_x(\tau) \delta \sigma(\tau) \right\rangle ds dt \right| \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon} \int_{\tau}^{\tau+\varepsilon} \left[\mathbb{E} |\mathbb{S}(t, \bar{x}(t), v) - \mathbb{S}(\tau, \bar{x}(\tau), v)|^2 \right]^{\frac{1}{2}} dt
\end{aligned}$$

$$(5.10) \quad + \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma_x(s) \delta \sigma(s) - \sigma_x(\tau) \delta \sigma(\tau) \right\rangle ds dt \right| = 0, \quad a.e. \quad \tau \in [0, T].$$

Then, by (5.7)–(5.10), it follows that, for a.e. $\tau \in [0, T]$,

$$(5.11) \quad \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt = \frac{1}{2} \partial_{\tau}^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \delta \sigma(\tau) \right) + \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \sigma_x(\tau) \delta \sigma(\tau) \right\rangle.$$

Combining (5.6) with (5.11), we obtain (5.3).

Step 2: In this step, we prove that, for a.e. $\tau \in [0, T]$,

$$(5.12) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v), y_2^{\varepsilon}(t) \right\rangle \chi_{E_{\varepsilon}}(t) dt = \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \delta b(\tau) \right\rangle.$$

Similar to (5.4), the explicit representation of $y_2^{\varepsilon}(\cdot)$ is given as follows:

$$(5.13) \quad y_2^{\varepsilon}(t) = \Phi(t) \int_0^t \Phi(s)^{-1} \left[\frac{1}{2} b_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) + \delta b(s) \chi_{E_{\varepsilon}}(s) - \frac{1}{2} \sigma_x(s) \sigma_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) - \sigma_x(s) \delta \sigma_x(s) y_1^{\varepsilon}(s) \chi_{E_{\varepsilon}}(s) \right] ds + \Phi(t) \int_0^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) + \delta \sigma_x(s) y_1^{\varepsilon}(s) \chi_{E_{\varepsilon}}(s) \right] dW(s).$$

Then,

$$(5.14) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v), y_2^{\varepsilon}(t) \right\rangle \chi_{E_{\varepsilon}}(t) dt \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \left[b_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) - \sigma_x(s) \sigma_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) \right] ds \right\rangle dt \\ &+ \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \delta b(s) \chi_{E_{\varepsilon}}(s) ds \right\rangle dt \\ &- \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \sigma_x(s) \delta \sigma_x(s) y_1^{\varepsilon}(s) \chi_{E_{\varepsilon}}(s) ds \right\rangle dt \\ &+ \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) + \delta \sigma_x(s) y_1^{\varepsilon}(s) \chi_{E_{\varepsilon}}(s) \right] dW(s) \right\rangle dt. \end{aligned}$$

By (5.4), $y_1^{\varepsilon}(t) = 0$ for any $t \in [0, \tau]$. Therefore, by Lemmas 3.2 and 4.2,

$$\lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{2\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \left[b_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) - \sigma_x(s) \sigma_{xx}(s) (y_1^{\varepsilon}(s), y_1^{\varepsilon}(s)) \right] ds \right\rangle dt \right|$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{2\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \left[b_{xx}(s)(y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \right. \right. \\
&\quad \left. \left. \left. - \sigma_x(s) \sigma_{xx}(s)(y_1^\varepsilon(s), y_1^\varepsilon(s)) \right] ds \right\rangle dt \right| \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon^2} \left\{ \int_{\tau}^{\tau+\varepsilon} \mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 dt \right\}^{\frac{1}{2}} \left\{ \int_{\tau}^{\tau+\varepsilon} \mathbb{E} \left| \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \cdot \right. \right. \\
&\quad \left. \left. \left[b_{xx}(s)(y_1^\varepsilon(s), y_1^\varepsilon(s)) - \sigma_x(s) \sigma_{xx}(s)(y_1^\varepsilon(s), y_1^\varepsilon(s)) \right] ds \right|^2 dt \right\}^{\frac{1}{2}} \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon^2} \left\{ \int_{\tau}^{\tau+\varepsilon} \mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 dt \right\}^{\frac{1}{2}} \cdot \\
&\quad \left\{ \int_{\tau}^{\tau+\varepsilon} (t - \tau)^2 \mathbb{E} \left[\sup_{s \in [0, T]} |\Phi(t) \Phi(s)^{-1}| \cdot \sup_{s \in [0, T]} |y_1^\varepsilon(s)|^2 \right]^2 dt \right\}^{\frac{1}{2}} \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon^{\frac{1}{2}}} \left\{ \int_{\tau}^{\tau+\varepsilon} \mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 dt \right\}^{\frac{1}{2}} \cdot \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^8 \right] \right\}^{\frac{1}{4}} \\
&= 0, \quad a.e. \quad \tau \in [0, T].
\end{aligned}
\tag{5.15}$$

Next, from Lemma 5.1 we conclude that

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \delta b(s) \chi_{E_\varepsilon}(s) ds \right\rangle dt \\
&= \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \delta b(\tau) \right\rangle \quad a.e. \quad \tau \in [0, T].
\end{aligned}
\tag{5.16}$$

Also, by Lemmas 3.2 and 4.2, we deduce that

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \sigma_x(s) \delta \sigma_x(s) y_1^\varepsilon(s) \chi_{E_\varepsilon}(s) ds \right\rangle dt \right| \\
&= \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \sigma_x(s) \delta \sigma_x(s) y_1^\varepsilon(s) ds \right\rangle dt \right| \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \left[\int_{\tau}^{\tau+\varepsilon} \mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 dt \right]^{\frac{1}{2}} \cdot \\
&\quad \left[\int_{\tau}^{\tau+\varepsilon} \mathbb{E} \left| \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \sigma_x(s) \delta \sigma_x(s) y_1^\varepsilon(s) ds \right|^2 dt \right]^{\frac{1}{2}} \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon^2} \left\{ \int_{\tau}^{\tau+\varepsilon} \mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 dt \right\}^{\frac{1}{2}} \cdot \\
&\quad \left\{ \int_{\tau}^{\tau+\varepsilon} (t - \tau)^2 \mathbb{E} \left[\sup_{s \in [0, T]} |\Phi(t) \Phi(s)^{-1}| \cdot \sup_{s \in [0, T]} |y_1^\varepsilon(s)| \right]^2 dt \right\}^{\frac{1}{2}} \\
&\leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon^{\frac{1}{2}}} \left\{ \int_{\tau}^{\tau+\varepsilon} \mathbb{E} |\mathbb{S}(t, \bar{x}(t), v)|^2 dt \right\}^{\frac{1}{2}} \cdot \left\{ \mathbb{E} \left[\sup_{s \in [0, T]} |y_1^\varepsilon(s)|^4 \right] \right\}^{\frac{1}{4}} \\
&= 0, \quad a.e. \quad \tau \in [0, T].
\end{aligned}
\tag{5.17}$$

In a similar way, we obtain that

$$\lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_0^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s)(y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \right. \right.$$

$$\begin{aligned}
& + \delta \sigma_x(s) y_1^\varepsilon(s) \chi_{E_\varepsilon}(s) \Big] dW(s) \Big\rangle dt \Big| \\
& = \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_\tau^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \right. \right. \\
& \quad \left. \left. \left. + \delta \sigma_x(s) y_1^\varepsilon(s) \right] dW(s) \right\rangle dt \right| \\
& \leq \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \Phi(t)^\top \mathbb{S}(t, \bar{x}(t), v) - \Phi(\tau)^\top \mathbb{S}(\tau, \bar{x}(\tau), v), \right. \right. \\
& \quad \left. \left. \int_\tau^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) + \delta \sigma_x(s) y_1^\varepsilon(s) \right] dW(s) \right\rangle dt \right| \\
& \quad + \lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon^2} \int_\tau^{\tau+\varepsilon} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \Phi(\tau) \int_\tau^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \right. \right. \\
& \quad \left. \left. \left. + \delta \sigma_x(s) y_1^\varepsilon(s) \right] dW(s) \right\rangle dt \right| \\
& \leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \left\{ \int_\tau^{\tau+\varepsilon} \mathbb{E} \left| \Phi(t)^\top \mathbb{S}(t, \bar{x}(t), v) - \Phi(\tau)^\top \mathbb{S}(\tau, \bar{x}(\tau), v) \right|^2 dt \right\}^{\frac{1}{2}} \cdot \\
& \quad \left\{ \int_\tau^{\tau+\varepsilon} \int_\tau^t \mathbb{E} \left| \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) + \delta \sigma_x(s) y_1^\varepsilon(s) \right] \right|^2 ds dt \right\}^{\frac{1}{2}} \\
& \leq \lim_{\varepsilon \rightarrow 0^+} \frac{C}{\varepsilon} \left\{ \int_\tau^{\tau+\varepsilon} \mathbb{E} \left| \Phi(t)^\top \mathbb{S}(t, \bar{x}(t), v) - \Phi(\tau)^\top \mathbb{S}(\tau, \bar{x}(\tau), v) \right|^2 dt \right\}^{\frac{1}{2}} \cdot \\
& \quad \left\{ \left[\mathbb{E} \left(\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^8 \right) \right]^{\frac{1}{2}} + \left[\mathbb{E} \left(\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^4 \right) \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\
& = 0, \quad a.e. \quad \tau \in [0, T].
\end{aligned}
\tag{5.18}$$

Here, we have used the fact that

$$\mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \Phi(\tau) \int_\tau^t \Phi(s)^{-1} \left[\frac{1}{2} \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) + \delta \sigma_x(s) y_1^\varepsilon(s) \right] dW(s) \right\rangle = 0,$$

for any $t \in [\tau, T]$.

Combining (5.15)–(5.18) with (5.14), we obtain (5.12).

Step 3: In this step, we prove that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{T}(t, \bar{x}(t), v) y_1^\varepsilon(t), y_1^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) dt = \frac{1}{2} \mathbb{E} \left\langle \mathbb{T}(\tau, \bar{x}(\tau), v) \delta \sigma(\tau), \delta \sigma(\tau) \right\rangle.
\tag{5.19}$$

Similar to the pervious discusses, we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{T}(t, \bar{x}(t), v) y_1^\varepsilon(t), y_1^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) dt \\
& = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{T}(t, \bar{x}(t), v) \left(-\Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_x(s) \delta \sigma(s) ds \right. \right. \\
& \quad \left. \left. + \Phi(t) \int_\tau^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right), \right. \\
& \quad \left. -\Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_x(s) \delta \sigma(s) ds + \Phi(t) \int_\tau^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{T}(t, \bar{x}(t), v) \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s), \right. \\
&\quad \left. \Phi(t) \int_{\tau}^t \Phi(s)^{-1} \delta \sigma(s) dW(s) \right\rangle dt \\
&= \frac{1}{2} \mathbb{E} \left\langle \mathbb{T}(\tau, \bar{x}(\tau), v) \delta \sigma(\tau), \delta \sigma(\tau) \right\rangle, \quad a.e. \quad \tau \in [0, T].
\end{aligned}$$

This proves (5.19).

Step 4: From Step 1–Step 3, we have proved that, for any $v \in V$,

$$\begin{aligned}
0 &\geq \limsup_{\varepsilon \rightarrow 0^+} \frac{J(\bar{u}(\cdot)) - J(u^\varepsilon(\cdot))}{\varepsilon^2} \\
&= \frac{1}{2} \mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \delta b(\tau) \right\rangle + \frac{1}{2} \partial_\tau^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \delta \sigma(\tau) \right) \\
&\quad + \frac{1}{4} \mathbb{E} \left\langle \mathbb{T}(\tau, \bar{x}(\tau), v) \delta \sigma(\tau), \delta \sigma(\tau) \right\rangle, \quad a.e. \quad \tau \in [0, T].
\end{aligned}$$

Therefore, for any $v \in V$, it follows that

$$\begin{aligned}
&\mathbb{E} \left\langle \mathbb{S}(\tau, \bar{x}(\tau), v), \delta b(\tau) \right\rangle + \partial_\tau^+ \left(\mathbb{S}(\tau, \bar{x}(\tau), v); \delta \sigma(\tau) \right) \\
&\quad + \frac{1}{2} \mathbb{E} \left\langle \mathbb{T}(\tau, \bar{x}(\tau), v) \delta \sigma(\tau), \delta \sigma(\tau) \right\rangle \leq 0 \quad a.e. \quad \tau \in [0, T].
\end{aligned}$$

This completes the proof of Theorem 4.3.

5.2. Proof of Theorem 4.4. We borrow some idea from the proof of [28, Theorem 3.9]. Denote by $\{t_i\}_{i=1}^\infty$ the totality of rational number in $[0, T]$, by $\{v^k\}_{k=1}^\infty$ a dense subset of V , and by $\{A_{ij}\}_{j=1}^\infty$ the countable subfamily of \mathcal{F}_{t_i} , $i \in \mathbb{N}$ such that for any $A \in \mathcal{F}_{t_i}$, there exists $\{A_{ij_n}\}_{n=1}^\infty \subset \{A_{ij}\}_{j=1}^\infty$ such that $\lim_{n \rightarrow \infty} P(A \Delta A_{ij_n}) = 0$, where $A \Delta A_{ij_n} = (A \setminus A_{ij_n}) \cup (A_{ij_n} \setminus A)$.

For any fixed t_i , v^k and $A_{ij} \in \mathcal{F}_{t_i}$, let $\tau \in [t_i, T]$, $\varepsilon \in (0, T - \tau)$, $E_\varepsilon = [\tau, \tau + \varepsilon]$, and write $u_{ij}^k(\omega, t) = \begin{cases} \bar{u}(\omega, t), & (\omega, t) \in (\Omega \times [0, T]) \setminus (A_{ij} \times [t_i, T]), \\ v^k, & (\omega, t) \in A_{ij} \times [t_i, T]. \end{cases}$ Clearly, $u_{ij}^k(\cdot) \in \mathcal{U}_{ad}$. Put $\hat{u}^\varepsilon(t) = \begin{cases} u_{ij}^k(t), & t \in E_\varepsilon, \\ \bar{u}(t), & t \in [0, T] \setminus E_\varepsilon. \end{cases}$ By Proposition 3.4 and using the condition (4.1), we have

$$\begin{aligned}
0 &\geq \frac{J(\bar{u}(\cdot)) - J(\hat{u}^\varepsilon(\cdot))}{\varepsilon^2} \\
&= \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left[\mathbb{H}(t, \bar{x}(t), v^k) + \langle \mathbb{S}(t, \bar{x}(t), v^k), \hat{y}_1^\varepsilon(t) + \hat{y}_2^\varepsilon(t) \rangle \right. \\
&\quad \left. + \frac{1}{2} \langle \mathbb{T}(t, \bar{x}(t), v^k) \hat{y}_1^\varepsilon(t), \hat{y}_1^\varepsilon(t) \rangle \right] \chi_{A_{ij}} \chi_{E_\varepsilon}(t) dt + o(1) \quad (\varepsilon \rightarrow 0^+) \\
&= \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left[\langle \mathbb{S}(t, \bar{x}(t), v^k), \hat{y}_1^\varepsilon(t) + \hat{y}_2^\varepsilon(t) \rangle \right. \\
(5.20) \quad &\quad \left. + \frac{1}{2} \langle \mathbb{T}(t, \bar{x}(t), v^k) \hat{y}_1^\varepsilon(t), \hat{y}_1^\varepsilon(t) \rangle \right] \chi_{A_{ij}} \chi_{E_\varepsilon}(t) dt + o(1) \quad (\varepsilon \rightarrow 0^+).
\end{aligned}$$

where $\hat{y}_1^\varepsilon(\cdot)$, $\hat{y}_2^\varepsilon(\cdot)$ are the solutions to the variational equations (3.4) and (3.5) with respect to $\hat{u}^\varepsilon(\cdot)$, respectively.

We first prove that there exists a sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$, $\varepsilon_\ell \rightarrow 0^+$ as $\ell \rightarrow \infty$, and,

$$(5.21) \quad \lim_{\ell \rightarrow \infty} \frac{1}{\varepsilon_\ell^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \hat{y}_1^{\varepsilon_\ell}(t) \right\rangle \chi_{A_{ij}} \chi_{E_{\varepsilon_\ell}}(t) dt \\ = \frac{1}{2} \mathbb{E} \left[\left\langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v^k), \sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right], \quad a.e. \tau \in [t_i, T],$$

By (5.4), $\hat{y}_1^\varepsilon(\cdot)$ enjoys the following explicit representation:

$$(5.22) \quad \hat{y}_1^\varepsilon(t) = -\Phi(t) \int_0^t \Phi(s)^{-1} \sigma_x(s) (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \chi_{E_\varepsilon}(s) ds \\ + \Phi(t) \int_0^t \Phi(s)^{-1} (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \chi_{E_\varepsilon}(s) dW(s).$$

Then,

$$(5.23) \quad \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \hat{y}_1^\varepsilon(t) \right\rangle \chi_{A_{ij}} \chi_{E_\varepsilon}(t) dt \\ = -\frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_x(s) \right. \\ \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} ds \right\rangle \chi_{A_{ij}} dt \\ + \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \Phi(t) \int_\tau^t \Phi(s)^{-1} \right. \\ \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt.$$

By Lemma 5.1, we obtain that for a.e. $\tau \in [t_i, T]$

$$(5.24) \quad \lim_{\varepsilon \rightarrow 0^+} \left[-\frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v), \Phi(t) \int_\tau^t \Phi(s)^{-1} \sigma_x(s) \right. \right. \\ \left. \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} ds \right\rangle \chi_{A_{ij}} dt \right] \\ = -\frac{1}{2} \mathbb{E} \left[\left\langle \mathbb{S}(\tau, \bar{x}(\tau), v^k), \sigma_x(\tau) (\sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))) \right\rangle \chi_{A_{ij}} \right].$$

On the other hand, by (4.4), we deduce that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \Phi(t) \int_\tau^t \Phi(s)^{-1} \right. \\ \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\ = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \Phi(\tau) \int_\tau^t \Phi(s)^{-1} \right. \\ \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\ + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_\tau^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \int_\tau^t b_x(s) \Phi(s) ds \int_\tau^t \Phi(s)^{-1} \right. \\ \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt$$

$$\begin{aligned}
& + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \int_{\tau}^t \sigma_x(s) \Phi(s) dW(s) \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt.
\end{aligned}
\tag{5.25}$$

Similar to the proof of (5.9)–(5.10), we obtain that, for a.e. $\tau \in [t_i, T]$,

$$\begin{aligned}
(5.26) \quad & \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \int_{\tau}^t b_x(s) \Phi(s) ds \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\
& = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \int_{\tau}^t \sigma_x(s) \Phi(s) dW(s) \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\
& = \frac{1}{2} \mathbb{E} \left[\left\langle \mathbb{S}(\tau, \bar{x}(\tau), v^k), \sigma_x(\tau) (\sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))) \right\rangle \chi_{A_{ij}} \right].
\end{aligned}
\tag{5.27}$$

Then, by (5.23)–(5.27), in order to prove (5.21), it remains to show that there exists a sequence $\{\varepsilon_{\ell}\}_{\ell=1}^{\infty}$, $\varepsilon_{\ell} \rightarrow 0^+$ as $\ell \rightarrow \infty$ such that

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \frac{1}{\varepsilon_{\ell}^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon_{\ell}} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \Phi(\tau) \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\
& = \frac{1}{2} \mathbb{E} \left[\left\langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v^k), \sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right], \quad a.e. \tau \in [t_i, T].
\end{aligned}
\tag{5.28}$$

By the regularity assumption (C3) and the Clark-Ocone representation formula, we have that

$$\begin{aligned}
& \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \Phi(\tau) \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\
& = \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \mathbb{E} \mathbb{S}(t, \bar{x}(t), v^k), \Phi(\tau) \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt \\
& + \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \left\langle \int_0^t \mathbb{E} [\mathcal{D}_s \mathbb{S}(t, \bar{x}(t), v^k) \mid \mathcal{F}_s] dW(s), \Phi(\tau) \int_{\tau}^t \Phi(s)^{-1} \right. \\
& \quad \left. \cdot (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} dW(s) \right\rangle \chi_{A_{ij}} dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \left\langle \mathcal{D}_s \mathbb{S}(t, \bar{x}(t), v^k), \right. \\
&\quad \left. \Phi(\tau) \Phi(s)^{-1} (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \right\rangle \chi_{A_{ij}} ds dt \\
&= \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \left\langle \mathcal{D}_s \mathbb{S}(t, \bar{x}(t), v^k) - \nabla \mathbb{S}(s, \bar{x}(s), v^k), \right. \\
&\quad \left. \Phi(\tau) \Phi(s)^{-1} (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \right\rangle \chi_{A_{ij}} ds dt \\
&\quad + \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \left\langle \nabla \mathbb{S}(s, \bar{x}(s), v^k), \right. \\
(5.29) \quad &\quad \left. \Phi(\tau) \Phi(s)^{-1} (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \right\rangle \chi_{A_{ij}} ds dt.
\end{aligned}$$

By the assumptions (C1)–(C3) and [28, Lemma 2.1], there exists a sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$, $\varepsilon_\ell \rightarrow 0^+$ as $\ell \rightarrow \infty$ such that

$$\begin{aligned}
&\lim_{\ell \rightarrow \infty} \frac{1}{\varepsilon_\ell^2} \left| \mathbb{E} \int_{\tau}^{\tau+\varepsilon_\ell} \int_{\tau}^t \left\langle \mathcal{D}_s \mathbb{S}(t, \bar{x}(t), v^k) - \nabla \mathbb{S}(s, \bar{x}(s), v^k), \right. \right. \\
&\quad \left. \left. \Phi(\tau) \Phi(s)^{-1} (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \right\rangle \chi_{A_{ij}} ds dt \right| \\
&\leq \lim_{\ell \rightarrow \infty} \frac{C}{\varepsilon_\ell} \left[\mathbb{E} \left(\sup_{s \in [\tau, T]} |\Phi(\tau) \Phi(s)^{-1}|^2 \right) \right]^{\frac{1}{2}} \\
&\quad \cdot \left[\mathbb{E} \int_{\tau}^{\tau+\varepsilon_\ell} \int_{\tau}^t \left| \mathcal{D}_s \mathbb{S}(t, \bar{x}(t), v^k) - \nabla \mathbb{S}(s, \bar{x}(s), v^k) \right|^2 ds dt \right]^{\frac{1}{2}} \\
&= 0, \quad a.e. \tau \in [t_i, T].
\end{aligned}$$

(5.30)

On the other hand, by Lemma 5.1,

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \int_{\tau}^t \left\langle \nabla \mathbb{S}(s, \bar{x}(s), v^k), \right. \\
&\quad \left. \Phi(\tau) \Phi(s)^{-1} (\sigma(s, \bar{x}(s), v^k) - \sigma(s, \bar{x}(s), \bar{u}(s))) \chi_{A_{ij}} \right\rangle \chi_{A_{ij}} ds dt \\
&= \frac{1}{2} \mathbb{E} \left[\left\langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v^k), \sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right], \quad a.e. \tau \in [t_i, T].
\end{aligned}$$

(5.31)

Combining (5.29), (5.30) with (5.31), we obtain (5.28). By (5.23)–(5.28), we obtain (5.21).

Next, similar to Steps 2 and 3 in the proof of Theorem 4.3, we obtain that

$$\begin{aligned}
(5.32) \quad &\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{S}(t, \bar{x}(t), v^k), \hat{y}_2^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) dt \\
&= \frac{1}{2} \mathbb{E} \left[\left\langle \mathbb{S}(\tau, \bar{x}(\tau), v^k), b(\tau, \bar{x}(\tau), v^k) - b(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right], \quad a.e. \tau \in [t_i, T].
\end{aligned}$$

and

$$(5.33) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \int_0^T \left\langle \mathbb{T}(t, \bar{x}(t), v), \hat{y}_1^\varepsilon(t), \hat{y}_1^\varepsilon(t) \right\rangle \chi_{E_\varepsilon}(t) dt$$

$$= \frac{1}{2} \mathbb{E} \left[\left\langle \mathbb{T}(\tau, \bar{x}(\tau), v^k) (\sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \right. \right. \\ \left. \left. \sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right], \quad a.e. \tau \in [t_i, T].$$

Finally, combining (5.20), (5.21), (5.32) and (5.33), we end up with

$$(5.34) \quad \begin{aligned} & \mathbb{E} \left[\left\langle \mathbb{S}(\tau, \bar{x}(\tau), v^k), b(\tau, \bar{x}(\tau), v^k) - b(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right] \\ & + \mathbb{E} \left[\left\langle \nabla \mathbb{S}(\tau, \bar{x}(\tau), v^k), \sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right] \\ & + \frac{1}{2} \mathbb{E} \left[\left\langle \mathbb{T}(\tau, \bar{x}(\tau), v^k) (\sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau))), \right. \right. \\ & \left. \left. \sigma(\tau, \bar{x}(\tau), v^k) - \sigma(\tau, \bar{x}(\tau), \bar{u}(\tau)) \right\rangle \chi_{A_{ij}} \right] \leq 0. \end{aligned}$$

By the arbitrariness of i, j, k , the construction of $\{A_{ij}\}_{i=1}^\infty$, the continuities of the filter \mathbb{F} and the map $v \mapsto \nabla \mathbb{S}(\tau, \bar{x}(\tau), v)$, and the density of $\{v^k\}_{k=1}^\infty$, we conclude that the desired necessary condition (4.7) holds. This completes the proof of Theorem 4.4.

Appendix A. Proof of Lemma 3.2. To simplify the notation, we only prove the 1-dimensional case (The high dimensional case can be proved in the same way).

The proof is long and requires heavy computations (The main idea comes from the proof of [26, Theorem 4.4, p. 128]). We will divide it into 4 steps

Step 1: Estimation of $\|y_1^\varepsilon\|_{\infty, \beta}^\beta, \|y_2^\varepsilon\|_{\infty, \beta}^\beta, \|y_3^\varepsilon\|_{\infty, \beta}^\beta$ and $\|y_4^\varepsilon\|_{\infty, \beta}^\beta$.

By the conditions (C1)–(C2) and the estimate (3.2), we have

$$(A.1) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta \right] \leq C \mathbb{E} \left[\int_0^T |\delta \sigma(t) \chi_{E_\varepsilon}(t)|^2 dt \right]^{\frac{\beta}{2}} \leq C \varepsilon^{\frac{\beta}{2}}.$$

In a similar way, we have

$$(A.2) \quad \begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^\beta \right] & \leq C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} b_{xx}(t) y_1^\varepsilon(t)^2 + \delta b(t) \chi_{E_\varepsilon}(t) \right| dt \right]^\beta \\ & \quad + C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} \sigma_{xx}(t) y_1^\varepsilon(t)^2 + \delta \sigma_x(t) y_1^\varepsilon(t) \chi_{E_\varepsilon}(t) \right|^2 dt \right]^{\frac{\beta}{2}} \\ & \leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{2\beta} \right] + \varepsilon^\beta \right. \\ & \quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{2\beta} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta \right] \varepsilon^{\frac{\beta}{2}} \right\} \leq C \varepsilon^\beta, \end{aligned}$$

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |y_3^\varepsilon(t)|^\beta \right] \\ & \leq C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} b_{xx}(t) (2y_1^\varepsilon(t) y_2^\varepsilon(t) + y_2^\varepsilon(t)^2) + \frac{1}{6} b_{xxx}(t) y_1^\varepsilon(t)^3 \right. \right. \\ & \quad \left. \left. + \delta b_x(t) y_1^\varepsilon(t) \chi_{E_\varepsilon}(t) \right| dt \right]^\beta + C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} \sigma_{xx}(t) (2y_1^\varepsilon(t) y_2^\varepsilon(t) + y_2^\varepsilon(t)^2) \right. \right. \\ & \quad \left. \left. + \frac{1}{6} \sigma_{xxx}(t) y_1^\varepsilon(t)^3 + \delta \sigma_x(t) y_2^\varepsilon(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \sigma_{xx}(t) y_1^\varepsilon(t)^2 \chi_{E_\varepsilon}(t) \right|^2 dt \right]^{\frac{\beta}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta |y_2^\varepsilon(t)|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^{2\beta} \right] \right. \\
&\quad + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{3\beta} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta \right] \varepsilon^\beta \\
&\quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^\beta \right] \varepsilon^{\frac{\beta}{2}} + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{2\beta} \right] \varepsilon^{\frac{\beta}{2}} \right\} \leq C \varepsilon^{\frac{3\beta}{2}},
\end{aligned}
\tag{A.3}$$

and

$$\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0, T]} |y_4^\varepsilon(t)|^\beta \right] \\
&\leq C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} b_{xx}(t) (2y_1^\varepsilon(t)y_3^\varepsilon(t) + 2y_2^\varepsilon(t)y_3^\varepsilon(t) + y_3^\varepsilon(t)^2) \right. \right. \\
&\quad + \frac{1}{6} b_{xxx}(t) (3y_1^\varepsilon(t)^2 y_2^\varepsilon(t) + 3y_1^\varepsilon(t)y_2^\varepsilon(t)^2 + y_2^\varepsilon(t)^3) \\
&\quad + \frac{1}{24} b_{xxxx}(t) y_1^\varepsilon(t)^4 + \delta b_x(t) y_2^\varepsilon(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta b_{xx}(t) y_1^\varepsilon(t)^2 \chi_{E_\varepsilon}(t) \left. \right| dt \Big]^\beta \\
&\quad + C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} \sigma_{xx}(t) (2y_1^\varepsilon(t)y_3^\varepsilon(t) + 2y_2^\varepsilon(t)y_3^\varepsilon(t) + y_3^\varepsilon(t)^2) \right. \right. \\
&\quad + \frac{1}{6} \sigma_{xxx}(t) (3y_1^\varepsilon(t)^2 y_2^\varepsilon(t) + 3y_1^\varepsilon(t)y_2^\varepsilon(t)^2 + y_2^\varepsilon(t)^3) \\
&\quad + \frac{1}{24} \sigma_{xxxx}(t) y_1^\varepsilon(t)^4 + \delta \sigma_x(t) y_3^\varepsilon(t) \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2} \delta \sigma_{xx}(t) (2y_1^\varepsilon(t)y_2^\varepsilon(t) + y_2^\varepsilon(t)^2) \chi_{E_\varepsilon}(t) + \frac{1}{6} \delta \sigma_{xxx}(t) y_1^\varepsilon(t)^3 \chi_{E_\varepsilon}(t) \left. \right|^2 dt \Big]^\beta \\
&\leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta |y_3^\varepsilon(t)|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^\beta |y_3^\varepsilon(t)|^\beta \right] \right. \\
&\quad + \mathbb{E} \left[\sup_{t \in [0, T]} |y_3^\varepsilon(t)|^{2\beta} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{2\beta} |y_2^\varepsilon(t)|^\beta \right] \\
&\quad + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta |y_2^\varepsilon(t)|^{2\beta} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^{3\beta} \right] \\
&\quad + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{4\beta} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^\beta \right] \varepsilon^\beta \\
&\quad + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{2\beta} \right] \varepsilon^\beta + \mathbb{E} \left[\sup_{t \in [0, T]} |y_3^\varepsilon(t)|^\beta \right] \varepsilon^{\frac{\beta}{2}} \\
&\quad + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta |y_2^\varepsilon(t)|^\beta \right] \varepsilon^{\frac{\beta}{2}} + \mathbb{E} \left[\sup_{t \in [0, T]} |y_2^\varepsilon(t)|^{2\beta} \right] \varepsilon^{\frac{\beta}{2}} \\
&\quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^{3\beta} \right] \varepsilon^{\frac{\beta}{2}} \right\} \leq C \varepsilon^{2\beta}
\end{aligned}
\tag{A.4}$$

Step 2: Estimation of $\|r_1\|_{\infty, \beta}^\beta$, $\|r_2\|_{\infty, \beta}^\beta$ and $\|r_3\|_{\infty, \beta}^\beta$.

By (3.3) and the condition (C1)–(C2), we have

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^\beta \right] &\leq C \mathbb{E} \left[\int_0^T |\delta b(t) \chi_{E_\varepsilon}(t)| dt \right]^\beta + C \mathbb{E} \left[\int_0^T |\delta \sigma(t) \chi_{E_\varepsilon}(t)|^2 dt \right]^\beta \\
&\leq C \varepsilon^\beta + C \varepsilon^{\frac{\beta}{2}} \leq C \varepsilon^{\frac{\beta}{2}}.
\end{aligned}
\tag{A.5}$$

Define

$$(A.6) \quad \begin{cases} \tilde{b}_x^\varepsilon(t) := \int_0^1 b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) d\theta, \\ \tilde{\sigma}_x^\varepsilon(t) := \int_0^1 \sigma_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) d\theta. \end{cases}$$

Then, $\delta x(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot)$ is the solution to the following stochastic differential equation:

$$(A.7) \quad \begin{cases} d\delta x(t) = \left[\tilde{b}_x^\varepsilon(t) \delta x(t) + \delta b(t) \chi_{E_\varepsilon}(t) \right] dt \\ \quad + \left[\tilde{\sigma}_x^\varepsilon(t) \delta x(t) + \delta \sigma(t) \chi_{E_\varepsilon}(t) \right] dW(t), \quad t \in [0, T], \\ \delta x(0) = 0. \end{cases}$$

Also, $r_1(\cdot) = \delta x(\cdot) - y_1^\varepsilon(\cdot)$ is the solution to the following stochastic differential equation:

$$(A.8) \quad \begin{cases} dr_1(t) = \left[\tilde{b}_x^\varepsilon(t) r_1(t) + \delta b(t) \chi_{E_\varepsilon}(t) + (\tilde{b}_x^\varepsilon(t) - b_x(t)) y_1^\varepsilon(t) \right] dt \\ \quad + \left[\tilde{\sigma}_x^\varepsilon(t) r_1(t) + (\tilde{\sigma}_x^\varepsilon(t) - \sigma_x(t)) y_1^\varepsilon(t) \right] dW(t), \quad t \in [0, T], \\ r_1(0) = 0. \end{cases}$$

Since

$$\begin{aligned} & \mathbb{E} \left[\int_0^T |\tilde{b}_x^\varepsilon(t) - b_x(t)| dt \right]^\beta \\ &= \mathbb{E} \left[\int_0^T \left| \int_0^1 b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) - b_x(t) d\theta \right| dt \right]^\beta \\ &\leq \mathbb{E} \left[\int_0^T \left(L |x^\varepsilon(t) - \bar{x}(t)| + |\delta b_x(t) \chi_{E_\varepsilon}(t)| \right) dt \right]^\beta \\ &\leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^\beta \right] + C \varepsilon^\beta \leq C \varepsilon^{\frac{\beta}{2}}, \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left[\int_0^T |\tilde{\sigma}_x^\varepsilon(t) - \sigma_x(t)|^2 dt \right]^{\frac{\beta}{2}} \\ &= \mathbb{E} \left[\int_0^T \left| \int_0^1 \sigma_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) - \sigma_x(t) d\theta \right|^2 dt \right]^{\frac{\beta}{2}} \\ &\leq C \mathbb{E} \left[\int_0^T \left(L |x^\varepsilon(t) - \bar{x}(t)|^2 + |\delta \sigma_x(t) \chi_{E_\varepsilon}(t)|^2 \right) dt \right]^{\frac{\beta}{2}} \\ &\leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^\beta \right] + C \varepsilon^{\frac{\beta}{2}} \leq C \varepsilon^{\frac{\beta}{2}}, \end{aligned}$$

we have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |r_1(t)|^\beta \right] &\leq C \mathbb{E} \left[\int_0^T |\delta b(t) \chi_{E_\varepsilon}(t) + (\tilde{b}_x^\varepsilon(t) - b_x(t)) y_1^\varepsilon(t)| dt \right]^\beta \\ &\quad + C \mathbb{E} \left[\int_0^T |(\tilde{\sigma}_x^\varepsilon(t) - \sigma_x(t)) y_1^\varepsilon(t)|^2 dt \right]^{\frac{\beta}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left\{ \varepsilon^\beta + \mathbb{E} \left[\left(\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta \right) \left(\int_0^T |\tilde{\sigma}_x^\varepsilon(t) - \sigma_x(t)| dt \right)^\beta \right] \right. \\
&\quad \left. + \mathbb{E} \left[\left(\sup_{t \in [0, T]} |y_1^\varepsilon(t)|^\beta \right) \left(\int_0^T |\tilde{b}_x^\varepsilon(t) - b_x(t)|^2 dt \right)^{\frac{\beta}{2}} \right] \right\} \leq C \varepsilon^\beta.
\end{aligned}
\tag{A.9}$$

This gives the estimation for $r_1(\cdot)$.

Next, we prove the estimation for $r_2(\cdot)$. For $\varphi = b, \sigma$, by Taylor's formula, we have

$$\begin{aligned}
&\varphi(t, x^\varepsilon(t), u^\varepsilon(t)) - \varphi(t, \bar{x}(t), \bar{u}(t)) \\
&= \varphi(t, x^\varepsilon(t), \bar{u}(t)) - \varphi(t, \bar{x}(t), \bar{u}(t)) + \varphi(t, \bar{x}(t), u^\varepsilon(t)) \\
&\quad - \varphi(t, \bar{x}(t), \bar{u}(t)) + \varphi(t, x^\varepsilon(t), u^\varepsilon(t)) - \varphi(t, \bar{x}(t), u^\varepsilon(t)) \\
&\quad - \varphi(t, x^\varepsilon(t), \bar{u}(t)) + \varphi(t, \bar{x}(t), \bar{u}(t)) \\
&= \varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \delta \varphi(t) \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2} \int_0^1 \theta^2 \varphi_{xxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \\
&\quad + \int_0^1 \left(\varphi_x(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
&\quad \left. - \varphi_x(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t) d\theta \\
&= \varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \frac{1}{6} \varphi_{xxx}(t) \delta x(t)^3 \\
&\quad + \delta \varphi(t) \chi_{E_\varepsilon}(t) + \delta \varphi_x(t) \delta x(t) \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 \varphi_{xxxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\
&\quad + \int_0^1 \theta \left(\varphi_{xx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
&\quad \left. - \varphi_{xx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^2 d\theta \\
&= \varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \frac{1}{6} \varphi_{xxx}(t) \delta x(t)^3 \\
&\quad + \delta \varphi(t) \chi_{E_\varepsilon}(t) + \delta \varphi_x(t) \delta x(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \varphi_{xx}(t) \delta x(t)^2 \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 \varphi_{xxxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\
&\quad + \frac{1}{2} \int_0^1 \theta^2 \left(\varphi_{xxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
&\quad \left. - \varphi_{xxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta \\
&= \varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \frac{1}{6} \varphi_{xxx}(t) \delta x(t)^3 + \delta \varphi(t) \chi_{E_\varepsilon}(t) \\
&\quad + \delta \varphi_x(t) \delta x(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \varphi_{xx}(t) \delta x(t)^2 \chi_{E_\varepsilon}(t) + \frac{1}{6} \delta \varphi_{xxx}(t) \delta x(t)^3 \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 \varphi_{xxxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta
\end{aligned}
\tag{A.10}$$

$$\begin{aligned}
&\varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \frac{1}{6} \varphi_{xxx}(t) \delta x(t)^3 + \delta \varphi(t) \chi_{E_\varepsilon}(t) \\
&\quad + \delta \varphi_x(t) \delta x(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \varphi_{xx}(t) \delta x(t)^2 \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 \varphi_{xxxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\
&\quad + \frac{1}{2} \int_0^1 \theta^2 \left(\varphi_{xxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
&\quad \left. - \varphi_{xxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta \\
&= \varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \frac{1}{6} \varphi_{xxx}(t) \delta x(t)^3 + \delta \varphi(t) \chi_{E_\varepsilon}(t) \\
&\quad + \delta \varphi_x(t) \delta x(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \varphi_{xx}(t) \delta x(t)^2 \chi_{E_\varepsilon}(t) + \frac{1}{6} \delta \varphi_{xxx}(t) \delta x(t)^3 \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 \varphi_{xxxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta
\end{aligned}
\tag{A.11}$$

$$\begin{aligned}
&\varphi_x(t) \delta x(t) + \frac{1}{2} \varphi_{xx}(t) \delta x(t)^2 + \frac{1}{6} \varphi_{xxx}(t) \delta x(t)^3 + \delta \varphi(t) \chi_{E_\varepsilon}(t) \\
&\quad + \delta \varphi_x(t) \delta x(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \varphi_{xx}(t) \delta x(t)^2 \chi_{E_\varepsilon}(t) + \frac{1}{6} \delta \varphi_{xxx}(t) \delta x(t)^3 \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 \varphi_{xxxx}(t, \theta \bar{x}(t) + (1 - \theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta
\end{aligned}
\tag{A.12}$$

$$(A.13) \quad +\frac{1}{6} \int_0^1 \theta^3 \left(\varphi_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\ \left. - \varphi_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^4 d\theta.$$

By (A.10) and (A.11), we find that $\delta x(\cdot)$ is the solution to the following differential equation:

$$(A.14) \quad \left\{ \begin{array}{l} d\delta x(t) = \left[b_x(t)\delta x(t) + \frac{1}{2}b_{xx}(t)\delta x(t)^2 + \delta b(t)\chi_{E_\varepsilon}(t) \right. \\ \quad + \frac{1}{2} \int_0^1 \theta^2 b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \\ \quad + \int_0^1 (b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad - b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t) d\theta \Big] dt \\ \quad + \left[\sigma_x(t)\delta x(t) + \frac{1}{2}\sigma_{xx}(t)\delta x(t)^2 \right. \\ \quad + \delta\sigma(t)\chi_{E_\varepsilon}(t) + \delta\sigma_x(t)\delta x(t)\chi_{E_\varepsilon}(t) \\ \quad + \frac{1}{2} \int_0^1 \theta^2 \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \\ \quad + \int_0^1 \theta (\sigma_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad - \sigma_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t)^2 d\theta \Big] dW(t), \\ \quad t \in [0, T], \\ \delta x(0) = 0. \end{array} \right.$$

Similarly, by (A.11)–(A.13), $\delta x(\cdot)$ is the solution to the stochastic differential equation

$$(A.15) \quad \left\{ \begin{array}{l} d\delta x(t) = \left[b_x(t)\delta x(t) + \frac{1}{2}b_{xx}(t)\delta x(t)^2 + \frac{1}{6}b_{xxx}(t)\delta x(t)^3 \right. \\ \quad + \delta b(t)\chi_{E_\varepsilon}(t) + \delta b_x(t)\delta x(t)\chi_{E_\varepsilon}(t) \\ \quad + \frac{1}{6} \int_0^1 \theta^3 b_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\ \quad + \int_0^1 \theta (b_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad - b_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t)^2 d\theta \Big] dt \\ \quad + \left[\sigma_x(t)\delta x(t) + \frac{1}{2}\sigma_{xx}(t)\delta x(t)^2 + \frac{1}{6}\sigma_{xxx}(t)\delta x(t)^3 \right. \\ \quad + \delta\sigma(t)\chi_{E_\varepsilon}(t) + \delta\sigma_x(t)\delta x(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta\sigma_{xx}(t)\delta x(t)^2\chi_{E_\varepsilon}(t) \\ \quad + \frac{1}{6} \int_0^1 \theta^3 \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\ \quad + \frac{1}{2} \int_0^1 \theta^2 (\sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad - \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t)^3 d\theta \Big] dW(t), \\ \quad t \in [0, T], \\ \delta x(0) = 0, \end{array} \right.$$

and the stochastic differential equation

$$(A.16) \left\{ \begin{array}{l} d\delta x(t) = \left[b_x(t)\delta x(t) + \frac{1}{2}b_{xx}(t)\delta x(t)^2 + \frac{1}{6}b_{xxx}(t)\delta x(t)^3 \right. \\ \quad + \delta b(t)\chi_{E_\varepsilon}(t) + \delta b_x(t)\delta x(t)\chi_{E_\varepsilon}(t) + \frac{1}{2}\delta b_{xx}(t)\delta x(t)^2\chi_{E_\varepsilon}(t) \\ \quad + \frac{1}{6}\int_0^1 \theta^3 b_{xxxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \\ \quad + \frac{1}{2}\int_0^1 \theta^2 (b_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad \quad \left. - b_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^3 d\theta \right] dt \\ \quad + \left[\sigma_x(t)\delta x(t) + \frac{1}{2}\sigma_{xx}(t)\delta x(t)^2 + \frac{1}{6}\sigma_{xxx}(t)\delta x(t)^3 \right. \\ \quad + \delta\sigma(t)\chi_{E_\varepsilon}(t) + \delta\sigma_x(t)\delta x(t)\chi_{E_\varepsilon}(t) \\ \quad + \frac{1}{2}\delta\sigma_{xx}(t)\delta x(t)^2\chi_{E_\varepsilon}(t) + \frac{1}{6}\delta\sigma_{xxx}(t)\delta x(t)^3\chi_{E_\varepsilon}(t) \\ \quad + \frac{1}{6}\int_0^1 \theta^3 \sigma_{xxxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \\ \quad + \frac{1}{6}\int_0^1 \theta^3 (\sigma_{xxxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad \quad \left. - \sigma_{xxxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^4 d\theta \right] dW(t), \\ \quad t \in [0, T], \\ \delta x(0) = 0. \end{array} \right.$$

Combining the variational equations (3.4) and (3.5) with the equation (A.14), we see that $r_2(\cdot)$ is the solution to the stochastic differential equation:

$$(A.17) \left\{ \begin{array}{l} dr_2(t) = \left[b_x(t)r_2(t) + \frac{1}{2}b_{xx}(t)(\delta x(t)^2 - y_1^\varepsilon(t)^2) \right. \\ \quad + \frac{1}{2}\int_0^1 \theta^2 b_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^3 d\theta \\ \quad + \int_0^1 (b_x(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad \quad \left. - b_x(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t) d\theta \right] dt \\ \quad + \left[\sigma_x(t)r_2(t) + \frac{1}{2}\sigma_{xx}(t)(\delta x(t)^2 - y_1^\varepsilon(t)^2) + \delta\sigma_x(t)r_1(t)\chi_{E_\varepsilon}(t) \right. \\ \quad + \frac{1}{2}\int_0^1 \theta^2 \sigma_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^3 d\theta \\ \quad + \int_0^1 \theta (\sigma_{xx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ \quad \quad \quad \left. - \sigma_{xx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^2 d\theta \right] dW(t), \\ \quad t \in [0, T], \\ r_2(0) = 0. \end{array} \right.$$

By the conditions (C1)–(C2), we have

$$(A.18) \quad \begin{aligned} & \mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^2 b_{xxx}(t, \theta\bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^3 d\theta \right| dt \right]^\beta \\ & \leq C \mathbb{E} \left[\int_0^T |\delta x(t)^3| dt \right]^\beta \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{3\beta} \right] \leq C\varepsilon^{\frac{3\beta}{2}}, \end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left| \int_0^1 \left(b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \right. \\
& \quad \left. \left. \left. - b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t) d\theta \right| dt \right]^\beta \\
(A.19) \quad & \leq C \mathbb{E} \left[\int_0^T |\delta x(t) \chi_{E_\varepsilon}(t)| dt \right]^\beta \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^\beta \right] \varepsilon^\beta \leq C \varepsilon^{\frac{3\beta}{2}},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^2 \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \right|^2 dt \right]^{\frac{\beta}{2}} \\
(A.20) \quad & \leq C \mathbb{E} \left[\int_0^T |\delta x(t)^3|^2 dt \right]^{\frac{\beta}{2}} \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{3\beta} \right] \leq C \varepsilon^{\frac{3\beta}{2}},
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left| \int_0^1 \theta (\sigma_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \\
& \quad \left. \left. - \sigma_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t)^2 d\theta \right|^2 dt \right]^{\frac{\beta}{2}} \\
(A.21) \quad & \leq C \mathbb{E} \left[\int_0^T |\delta x(t)^2 \chi_{E_\varepsilon}(t)|^2 dt \right]^{\frac{\beta}{2}} \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{2\beta} \right] \varepsilon^{\frac{\beta}{2}} \leq C \varepsilon^{\frac{3\beta}{2}}.
\end{aligned}$$

On the other hand, combining (A.1) and (A.5) with (A.9), we have

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - y_1^\varepsilon(t)^2|^\beta \right] = \mathbb{E} \left[\sup_{t \in [0, T]} (|r_1(t)|^\beta |\delta x(t) + y_1^\varepsilon(t)|^\beta) \right] \\
(A.22) \quad & \leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |r_1(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t) + y_1^\varepsilon(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \leq C \varepsilon^{\frac{3\beta}{2}}.
\end{aligned}$$

Then, combining (A.5), (A.18)–(A.22) with (A.17), we obtain that

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |r_2(t)|^\beta \right] \\
& \leq C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} b_{xx}(t) (\delta x(t)^2 - y_1^\varepsilon(t)^2) \right. \right. \\
& \quad + \frac{1}{2} \int_0^1 \theta^2 b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \\
& \quad + \int_0^1 (b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\
& \quad \quad \left. - b_x(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t) d\theta \right| dt \right]^\beta \\
& + C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} \sigma_{xx}(t) (\delta x(t)^2 - y_1^\varepsilon(t)^2) + \delta \sigma_x(t) r_1(t) \chi_{E_\varepsilon}(t) \right. \right. \\
& \quad + \frac{1}{2} \int_0^1 \theta^2 \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^3 d\theta \\
& \quad \left. + \int_0^1 \theta (\sigma_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right.
\end{aligned}$$

$$\begin{aligned}
& -\sigma_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^2 d\theta \Big|^2 dt \Big]^{\frac{\beta}{2}} \\
& \leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - y_1^\varepsilon(t)^2|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{3\beta} \right] \right. \\
(A.23) \quad & \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^\beta \right] \varepsilon^\beta + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{2\beta} \right] \varepsilon^{\frac{\beta}{2}} \right\} \leq C \varepsilon^{\frac{3\beta}{2}}.
\end{aligned}$$

This proves the estimation for $r_2(\cdot)$.

Now, we prove the estimate for $r_3(\cdot)$.

Combining the variational equation (3.4), (3.5) and (3.6) with (A.15), we see that, $r_3(\cdot)$ is the solution to the stochastic differential equation:

$$(A.24) \quad \left\{ \begin{aligned} dr_3(t) = & \left[b_x(t)r_3(t) + \frac{1}{2}b_{xx}(t)(\delta x(t)^2 - \gamma(t)^2) \right. \\ & + \frac{1}{6}b_{xxx}(t)(\delta x(t)^3 - y_1^\varepsilon(t)^3) + \delta b_x(t)r_1(t)\chi_{E_\varepsilon}(t) \\ & + \frac{1}{6}\int_0^1 \theta^3 b_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \\ & + \int_0^1 \theta (b_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ & - b_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^2 d\theta \Big] dt \\ & + \left[\sigma_x(t)r_3(t) + \frac{1}{2}\sigma_{xx}(t)(\delta x(t)^2 - \gamma(t)^2) \right. \\ & + \frac{1}{6}\sigma_{xxx}(t)(\delta x(t)^3 - y_1^\varepsilon(t)^3) + \delta \sigma_x(t)r_2(t)\chi_{E_\varepsilon}(t) \\ & + \frac{1}{2}\delta \sigma_{xx}(t)(\delta x(t)^2 - y_1^\varepsilon(t)^2)\chi_{E_\varepsilon}(t) \\ & + \frac{1}{6}\int_0^1 \theta^3 \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \\ & + \frac{1}{2}\int_0^1 \theta^2 (\sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\ & - \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^3 d\theta \Big] dW(t), \\ & \left. r_3(0) = 0. \right\}
\end{aligned} \right.$$

Similar to (A.18), we can prove that

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^3 b_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \right| dt \right]^\beta \\
(A.25) \quad & \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{4\beta} \right] \leq C \varepsilon^{2\beta}.
\end{aligned}$$

Similar to (A.19), we have

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^2 \{ \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \\
& \quad \left. \left. - \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \} \delta x(t)^3 d\theta \right| dt \right]^\beta \\
(A.26) \quad & \leq C \mathbb{E} \left[\int_0^T |\delta x(t)^3 \chi_{E_\varepsilon}(t)| dt \right]^\beta \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{3\beta} \right] \varepsilon^{\frac{\beta}{2}} \leq C \varepsilon^{2\beta}.
\end{aligned}$$

In a similar way, we have

$$\mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^3 \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \right|^2 dt \right]^{\frac{\beta}{2}}$$

$$(A.27) \quad \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{4\beta} \right] \leq C \varepsilon^{2\beta},$$

and

$$(A.28) \quad \begin{aligned} & \mathbb{E} \left(\int_0^T \left| \int_0^1 \theta^2 \{ \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \\ & \quad \left. \left. - \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \} \delta x(t)^3 d\theta \right|^2 dt \right)^{\frac{\beta}{2}} \\ & \leq C \mathbb{E} \left(\int_0^T |\delta x(t)^3 \chi_{E_\varepsilon}(t)|^2 dt \right)^{\frac{\beta}{2}} \leq C \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{3\beta} \right] \varepsilon^{\frac{\beta}{2}} \leq C \varepsilon^{2\beta}. \end{aligned}$$

On the other hand, by (A.1), (A.2), (A.5) and (A.23), we get that

$$(A.29) \quad \begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \gamma(t)^2|^\beta \right] \\ &= \mathbb{E} \left[\sup_{t \in [0, T]} \left(|r_2(t)|^\beta \cdot |\delta x(t) + y_1^\varepsilon(t) + y_2^\varepsilon(t)|^\beta \right) \right] \\ &\leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |r_2(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \cdot \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t) + y_1^\varepsilon(t) + y_2^\varepsilon(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{2\beta}. \end{aligned}$$

Also, by (A.1), (A.5) and (A.9), we have

$$(A.30) \quad \begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x^3 - y_1^\varepsilon(t)^3|^\beta \right] \\ &= \mathbb{E} \left[\sup_{t \in [0, T]} \left(|r_1(t)|^\beta \cdot |\delta x(t)^2 + \delta x(t)y_1^\varepsilon(t) + y_1^\varepsilon(t)^2|^\beta \right) \right] \\ &\leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |r_1(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \cdot \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 + \delta x(t)y_1^\varepsilon(t) + y_1^\varepsilon(t)^2|^{2\beta} \right] \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{2\beta}. \end{aligned}$$

Combining (A.25)–(A.30) with (3.2), we obtain that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |r_3(t)|^\beta \right] \\ &\leq C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} b_{xx}(t) (\delta x(t)^2 - \gamma(t)^2) \right. \right. \\ & \quad + \frac{1}{6} b_{xxx}(t) (\delta x(t)^3 - y_1^\varepsilon(t)^3) + \delta b_x(t) r_1(t) \chi_{E_\varepsilon}(t) \\ & \quad + \frac{1}{6} \int_0^1 \theta^3 b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\ & \quad + \int_0^1 \theta \left(b_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\ & \quad \left. \left. - b_{xx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^2 d\theta \right| dt \Big]^\beta \\ & \quad + C \mathbb{E} \left[\int_0^T \left| \frac{1}{2} \sigma_{xx}(t) (\delta x(t)^2 - \gamma(t)^2) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6} \sigma_{xxx}(t) (\delta x(t)^3 - y_1^\varepsilon(t)^3) + \delta \sigma_x(t) r_2(t) \chi_{E_\varepsilon}(t) \\
& + \frac{1}{2} \delta \sigma_{xx}(t) (\delta x(t)^2 - y_1^\varepsilon(t)^2) \chi_{E_\varepsilon}(t) \\
& + \frac{1}{6} \int_0^1 \theta^3 \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \\
& + \frac{1}{2} \int_0^1 \theta^2 \left(\sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
& \quad \left. - \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta \Big|^2 dt \Big]^{\frac{\beta}{2}} \\
& \leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \gamma(t)^2|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^3 - y_1^\varepsilon(t)^3|^\beta \right] \right. \\
& \quad + \mathbb{E} \left[\sup_{t \in [0, T]} |r_1(t)|^\beta \right] \varepsilon^\beta + \varepsilon^{2\beta} + \varepsilon^{2\beta} + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \gamma(t)^2|^\beta \right] \\
& \quad + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^3 - y_1^\varepsilon(t)^3|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |r_2(t)|^\beta \right] \varepsilon^{\frac{\beta}{2}} \\
& \quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - y_1^\varepsilon(t)^2|^\beta \right] \varepsilon^{\frac{\beta}{2}} + \varepsilon^{2\beta} + \varepsilon^{2\beta} \right\} \leq C \varepsilon^{2\beta}.
\end{aligned} \tag{A.31}$$

This proves the estimate for $r_3(\cdot)$.

Step 3: We now estimate $\|(\delta x)^2 - \eta^2\|_{\infty, \beta}^\beta$, $\|(\delta x)^3 - \gamma^3\|_{\infty, \beta}^\beta$ and $\|(\delta x)^4 - (y_1^\varepsilon)^4\|_{\infty, \beta}^\beta$. First, by (A.1)–(A.3), (A.5) and (A.31), we have

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \eta(t)^2|^\beta \right] \\
& = \mathbb{E} \left[\sup_{t \in [0, T]} \left(|r_3(t)|^\beta \cdot |\delta x(t) + y_1^\varepsilon(t) + y_2^\varepsilon(t) + y_3^\varepsilon(t)|^\beta \right) \right] \\
& \leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |r_3(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t) + y_1^\varepsilon(t) + y_2^\varepsilon(t) + y_3^\varepsilon(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \\
& \leq C \varepsilon^{\frac{5\beta}{2}}.
\end{aligned} \tag{A.32}$$

Next, by (A.1), (A.2), (A.5) and (A.23), we get

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x^3 - \gamma(t)^3|^\beta \right] \\
& = \mathbb{E} \left[\sup_{t \in [0, T]} \left(|r_2(t)|^\beta \cdot |\delta x(t)^2 + \delta x(t)\gamma(t) + \gamma(t)^2|^\beta \right) \right] \\
& \leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |r_2(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \cdot \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 + \delta x(t)\gamma(t) + \gamma(t)^2|^{2\beta} \right] \right)^{\frac{1}{2}} \\
& \leq C \varepsilon^{\frac{5\beta}{2}}.
\end{aligned} \tag{A.33}$$

Finally, by (A.1), (A.5) and (A.9), we have

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x^4 - (y_1^\varepsilon)^4|^\beta \right] \\
& = \mathbb{E} \left[\sup_{t \in [0, T]} \left(|r_1(t)|^\beta \cdot |\delta x(t) + y_1^\varepsilon(t)|^\beta \cdot |\delta x(t)^2 + y_1^\varepsilon(t)^2|^\beta \right) \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\mathbb{E} \left[\sup_{t \in [0, T]} |r_1(t)|^{2\beta} \right] \right)^{\frac{1}{2}} \cdot \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t) + y_1^\varepsilon(t)|^{4\beta} \right] \right)^{\frac{1}{4}} \cdot \\
&\quad \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 + y_1^\varepsilon(t)^2|^{4\beta} \right] \right)^{\frac{1}{4}} \\
\text{(A.34)} \quad &\leq C\varepsilon^{\frac{5\beta}{2}}.
\end{aligned}$$

Step 4: Estimate for $\|r_4\|_{\infty, \beta}^\beta$.

By (3.4)–(3.7) and (A.16), we obtain that

$$\begin{aligned}
\text{(A.35)} \quad &\left\{ \begin{aligned}
dr_4(t) = & \left[b_x(t)r_4(t) + \frac{1}{2}b_{xx}(t)(\delta x(t)^2 - \eta(t)^2) \right. \\
& + \frac{1}{6}b_{xxx}(t)(\delta x(t)^3 - \gamma(t)^3) + \delta b_x(t)r_2(t)\chi_{E_\varepsilon}(t) \\
& + \frac{1}{2}\delta b_{xx}(t)(\delta x(t)^2 - y_1^\varepsilon(t)^2)\chi_{E_\varepsilon}(t) \\
& + \frac{1}{6}\int_0^1 \theta^3 b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \\
& - \frac{1}{24}b_{xxxx}(t)y_1^\varepsilon(t)^4 \\
& + \frac{1}{2}\int_0^1 \theta^2 (b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\
& \quad \left. - b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^3 d\theta \right] dt \\
& + \left[\sigma_x(t)r_4(t) + \frac{1}{2}\sigma_{xx}(t)(\delta x(t)^2 - \eta(t)^2) \right. \\
& + \frac{1}{6}\sigma_{xxx}(t)(\delta x(t)^3 - \gamma(t)^3) + \delta \sigma_x(t)r_3(t)\chi_{E_\varepsilon}(t) \\
& + \frac{1}{2}\delta \sigma_{xx}(t)(\delta x(t)^2 - \gamma(t)^2)\chi_{E_\varepsilon}(t) \\
& + \frac{1}{6}\delta \sigma_{xxx}(t)(\delta x(t)^3 - y_1^\varepsilon(t)^3)\chi_{E_\varepsilon}(t) \\
& + \frac{1}{6}\int_0^1 \theta^3 \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \\
& - \frac{1}{24}\sigma_{xxxx}(t)y_1^\varepsilon(t)^4 \\
& + \frac{1}{6}\int_0^1 \theta^3 (\sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\
& \quad \left. - \sigma_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)))\delta x(t)^4 d\theta \right] dW(t), \\
& t \in [0, T], \\
& r_4(0) = 0.
\end{aligned} \right.
\end{aligned}$$

By (A.34) and the conditions (C1)–(C2), we have

$$\begin{aligned}
&\mathbb{E} \left[\int_0^T \left| \frac{1}{6} \int_0^1 \theta^3 b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))\delta x(t)^4 d\theta \right. \right. \\
&\quad \left. \left. - \frac{1}{24}b_{xxxx}(t)y_1^\varepsilon(t)^4 \right| dt \right]^\beta \\
&\leq C\mathbb{E} \left[\frac{1}{6} \int_0^T \left| \int_0^1 \theta^3 (b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right. \right. \\
&\quad \left. \left. - b_{xxx}(t))\delta x(t)^4 d\theta \right| dt \right]^\beta \\
&\quad + C\mathbb{E} \left[\frac{1}{6} \int_0^T \left| \int_0^1 \theta^3 b_{xxx}(t)\delta x(t)^4 d\theta - \frac{1}{4}b_{xxxx}(t)y_1^\varepsilon(t)^4 \right| dt \right]^\beta
\end{aligned}$$

$$\begin{aligned}
&\leq C\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^{5\beta}| dt \right] + C\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^4 - y_1^\varepsilon(t)^4|^\beta \right] \\
(A.36) \quad &\leq C\varepsilon^{\frac{5\beta}{2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{E} \left[\int_0^T \left| \frac{1}{6} \int_0^1 \theta^3 \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta \right. \right. \\
&\quad \left. \left. - \frac{1}{24} \sigma_{xxxx}(t) y_1^\varepsilon(t)^4 \right|^2 dt \right]^{\frac{\beta}{2}} \\
(A.37) \quad &\leq C\varepsilon^{\frac{5\beta}{2}}.
\end{aligned}$$

Next, similar to (A.19), we have

$$\begin{aligned}
&\mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^2 \left(b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \right. \\
&\quad \left. \left. \left. - b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta \right| dt \right]^\beta \\
(A.38) \quad &\leq C\mathbb{E} \left[\int_0^T |\delta x(t)^3 \chi_{E_\varepsilon}(t)| dt \right]^\beta \leq C\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{3\beta} \right] \varepsilon^\beta \leq C\varepsilon^{\frac{5\beta}{2}}.
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E} \left[\int_0^T \left| \int_0^1 \theta^3 \left(\sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \right. \right. \\
&\quad \left. \left. \left. - \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^4 d\theta \right|^2 dt \right]^{\frac{\beta}{2}} \\
(A.39) \quad &\leq C\mathbb{E} \left[\int_0^T |\delta x(t)^4 \chi_{E_\varepsilon}(t)|^2 dt \right]^{\frac{\beta}{2}} \leq C\mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)|^{4\beta} \right] \varepsilon^{\frac{\beta}{2}} \leq C\varepsilon^{\frac{5\beta}{2}}.
\end{aligned}$$

Finally, by (3.2) and (A.32)–(A.39), we obtain that

$$\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0, T]} |r_4(t)|^\beta \right] \\
&\leq C\mathbb{E} \left[\int_0^T \left| \frac{1}{2} b_{xx}(t) (\delta x(t)^2 - \eta(t)^2) + \frac{1}{6} b_{xxx}(t) (\delta x(t)^3 - \gamma(t)^3) \right. \right. \\
&\quad + \delta b_x(t) r_2(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta b_{xx}(t) (\delta x(t)^2 - y_1^\varepsilon(t)^2) \chi_{E_\varepsilon}(t) \\
&\quad + \frac{1}{6} \int_0^1 \theta^3 b_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta - \frac{1}{24} b_{xxxx}(t) y_1^\varepsilon(t)^4 \\
&\quad + \frac{1}{2} \int_0^1 \theta^2 \left(b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \right. \\
&\quad \left. \left. - b_{xxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \right) \delta x(t)^3 d\theta \right| dt \right]^\beta \\
&\quad + C\mathbb{E} \left[\int_0^T \left| \frac{1}{2} \sigma_{xx}(t) (\delta x(t)^2 - \eta(t)^2) + \frac{1}{6} \sigma_{xxx}(t) (\delta x(t)^3 - \gamma(t)^3) \right. \right. \\
&\quad \left. \left. + \delta \sigma_x(t) r_3(t) \chi_{E_\varepsilon}(t) + \frac{1}{2} \delta \sigma_{xx}(t) (\delta x(t)^2 - \gamma(t)^2) \chi_{E_\varepsilon}(t) \right| dt \right]^\beta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6} \delta \sigma_{xxx}(t) (\delta x(t)^3 - y_1^\varepsilon(t)^3) \chi_{E_\varepsilon}(t) \\
& + \frac{1}{6} \int_0^1 \theta^3 \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t)) \delta x(t)^4 d\theta - \frac{1}{24} \sigma_{xxxx}(t) y_1^\varepsilon(t)^4 \\
& + \frac{1}{6} \int_0^1 \theta^3 (\sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), u^\varepsilon(t)) \\
& \quad - \sigma_{xxxx}(t, \theta \bar{x}(t) + (1-\theta)x^\varepsilon(t), \bar{u}(t))) \delta x(t)^4 d\theta \Big|^\beta dt \Big]^{\frac{\beta}{2}} \\
& \leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \eta(t)^2|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^3 - \gamma(t)^3|^\beta \right] \right. \\
& \quad + \mathbb{E} \left[\sup_{t \in [0, T]} |r_2(t)|^\beta \right] \varepsilon^\beta + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - y_1^\varepsilon(t)^2|^\beta \right] \varepsilon^\beta \\
& \quad + \varepsilon^{\frac{5\beta}{2}} + \varepsilon^{\frac{5\beta}{2}} + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \eta(t)^2|^\beta \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^3 - \gamma(t)^3|^\beta \right] \\
& \quad + \mathbb{E} \left[\sup_{t \in [0, T]} |r_3(t)|^\beta \right] \varepsilon^{\frac{\beta}{2}} + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^2 - \gamma(t)^2|^\beta \right] \varepsilon^{\frac{\beta}{2}} \\
& \quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |\delta x(t)^3 - y_1^\varepsilon(t)^3|^\beta \right] \varepsilon^{\frac{\beta}{2}} + \varepsilon^{\frac{5\beta}{2}} + \varepsilon^{\frac{5\beta}{2}} \right\} \\
& \leq C \varepsilon^{\frac{5\beta}{2}}.
\end{aligned}$$

This completes the proof of Lemma 3.2.

Appendix B. Proof of Proposition 3.4. First, by (3.4)–(3.7), we have

$$\begin{aligned}
\xi(t) = & \int_0^t \left[b_x(s) \xi(s) + \frac{1}{2} b_{xx}(s) (\eta(s), \eta(s)) + \frac{1}{6} b_{xxx}(s) (\gamma(s), \gamma(s), \gamma(s)) \right. \\
& + \frac{1}{24} b_{xxxx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s)) + \delta b(s) \chi_{E_\varepsilon}(s) \\
& + \delta b_x(s) \gamma(s) \chi_{E_\varepsilon}(s) + \left. \frac{1}{2} \delta b_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) \chi_{E_\varepsilon}(s) \right] ds \\
& + \int_0^t \left[\sigma_x(s) \xi(s) + \frac{1}{2} \sigma_{xx}(s) (\eta(s), \eta(s)) + \frac{1}{6} \sigma_{xxx}(s) (\gamma(s), \gamma(s), \gamma(s)) \right. \\
& + \frac{1}{24} \sigma_{xxxx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s)) + \delta \sigma(s) \chi_{E_\varepsilon}(s) + \delta \sigma_x(s) \eta(s) \chi_{E_\varepsilon}(s) \\
\text{(B.1)} \quad & \left. + \frac{1}{2} \delta \sigma_{xx}(s) (\gamma(s), \gamma(s)) \chi_{E_\varepsilon}(s) + \frac{1}{6} \delta \sigma_{xxx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s)) \chi_{E_\varepsilon}(s) \right] dW(s),
\end{aligned}$$

$$\begin{aligned}
\eta(t) = & \int_0^t \left[b_x(s) \eta(s) + \frac{1}{2} b_{xx}(s) (\gamma(s), \gamma(s)) + \frac{1}{6} b_{xxx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \\
& + \delta b(s) \chi_{E_\varepsilon}(s) + \delta b_x(s) y_1^\varepsilon(s) \chi_{E_\varepsilon}(s) \left. \right] ds \\
& + \int_0^t \left[\sigma_x(s) \eta(s) + \frac{1}{2} \sigma_{xx}(s) (\gamma(s), \gamma(s)) + \frac{1}{6} \sigma_{xxx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \\
\text{(B.2)} \quad & \left. + \delta \sigma(s) \chi_{E_\varepsilon}(s) + \delta \sigma_x(s) \gamma(s) \chi_{E_\varepsilon}(s) + \frac{1}{2} \delta \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) \chi_{E_\varepsilon}(s) \right] dW(s),
\end{aligned}$$

and

$$\gamma(t) = \int_0^t \left[b_x(s) \gamma(s) + \frac{1}{2} b_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) + \delta b(s) \chi_{E_\varepsilon}(s) \right] ds$$

$$(B.3) \quad + \int_0^t \left[\sigma_x(s) \gamma(s) + \frac{1}{2} \sigma_{xx}(s) (y_1^\varepsilon(s), y_1^\varepsilon(s)) \right. \\ \left. + \delta \sigma(s) \chi_{E_\varepsilon}(s) + \delta \sigma_x(s) y_1^\varepsilon(s) \chi_{E_\varepsilon}(s) \right] dW(s).$$

Then, using the formula (2.1) in Lemma 2.1, we obtain that

$$(B.4) \quad \begin{aligned} & \mathbb{E} \langle h_x(\bar{x}(T)), \xi(T) \rangle = -\mathbb{E} \langle p_1(T), \xi(T) \rangle \\ & = -\mathbb{E} \int_0^T \left[\frac{1}{2} \langle p_1(t), b_{xx}(t) (\eta(t), \eta(t)) \rangle + \frac{1}{2} \langle q_1(t), \sigma_{xx}(t) (\eta(t), \eta(t)) \rangle \right. \\ & \quad + \frac{1}{6} \langle p_1(t), b_{xxx}(t) (\gamma(t), \gamma(t), \gamma(t)) \rangle + \frac{1}{6} \langle q_1(t), \sigma_{xxx}(t) (\gamma(t), \gamma(t), \gamma(t)) \rangle \\ & \quad + \frac{1}{24} \langle p_1(t), b_{xxxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\ & \quad + \frac{1}{24} \langle q_1(t), \sigma_{xxxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\ & \quad + \langle f_x(t), \xi(t) \rangle + \langle p_1(t), \delta b(t) \rangle \chi_{E_\varepsilon}(t) + \langle q_1(t), \delta \sigma(t) \rangle \chi_{E_\varepsilon}(t) \\ & \quad + \langle p_1(t), \delta b_x(t) \gamma(t) \rangle \chi_{E_\varepsilon}(t) + \langle q_1(t), \delta \sigma_x(t) \eta(t) \rangle \chi_{E_\varepsilon}(t) \\ & \quad + \frac{1}{2} \langle p_1(t), \delta b_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \chi_{E_\varepsilon}(t) \\ & \quad \left. + \frac{1}{2} \langle q_1(t), \delta \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \chi_{E_\varepsilon}(t) \right] dt + o(\varepsilon^2), \quad (\varepsilon \rightarrow 0^+), \end{aligned}$$

$$\begin{aligned} & \mathbb{E} \langle h_{xx}(\bar{x}(T)) \eta(T), \eta(T) \rangle = -\mathbb{E} \langle p_2(T) \eta(T), \eta(T) \rangle \\ & = -\mathbb{E} \int_0^T \left[\frac{1}{2} \langle p_2(t) b_{xx}(t) (\gamma(t), \gamma(t)), \eta(t) \rangle + \frac{1}{2} \langle p_2(t) \eta(t), b_{xx}(t) (\gamma(t), \gamma(t)) \rangle \right. \\ & \quad + \frac{1}{2} \langle q_2(t) \sigma_{xx}(t) (\gamma(t), \gamma(t)), \eta(t) \rangle + \frac{1}{2} \langle q_2(t) \eta(t), \sigma_{xx}(t) (\gamma(t), \gamma(t)) \rangle \\ & \quad + \frac{1}{6} \langle p_2(t) b_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)), \eta(t) \rangle \\ & \quad + \frac{1}{6} \langle p_2(t) \eta(t), b_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\ & \quad + \frac{1}{6} \langle q_2(t) \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)), \eta(t) \rangle \\ & \quad + \frac{1}{6} \langle q_2(t) \eta(t), \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\ & \quad + \langle p_2(t) \delta b(t), \eta(t) \rangle \chi_{E_\varepsilon}(t) + \langle p_2(t) \eta(t), \delta b(t) \rangle \chi_{E_\varepsilon}(t) \\ & \quad + \langle q_2(t) \delta \sigma(t), \eta(t) \rangle \chi_{E_\varepsilon}(t) + \langle q_2(t) \eta(t), \delta \sigma(t) \rangle \chi_{E_\varepsilon}(t) \\ & \quad + \langle p_2(t) \delta b_x(t) y_1^\varepsilon(t), \eta(t) \rangle \chi_{E_\varepsilon}(t) + \langle p_2(t) \eta(t), \delta b_x(t) y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) \\ & \quad + \langle q_2(t) \delta \sigma_x(t) \gamma(t), \eta(t) \rangle \chi_{E_\varepsilon}(t) + \langle q_2(t) \eta(t), \delta \sigma_x(t) \gamma(t) \rangle \chi_{E_\varepsilon}(t) \\ & \quad + \langle p_2(t) \sigma_x(t) \eta(t), \frac{1}{2} \sigma_{xx}(t) (\gamma(t), \gamma(t)) + \frac{1}{6} \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\ & \quad + \langle p_2(t) \left[\frac{1}{2} \sigma_{xx}(t) (\gamma(t), \gamma(t)) + \frac{1}{6} \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \right], \sigma_x(t) \eta(t) \rangle \\ & \quad \left. + \langle p_2(t) \sigma_x(t) \eta(t), \delta \sigma(t) + \delta \sigma_x(t) \gamma(t) \rangle \chi_{E_\varepsilon}(t) \right] \end{aligned}$$

$$\begin{aligned}
& + \langle p_2(t)(\delta\sigma(t) + \delta\sigma_x(t)\gamma(t)), \sigma_x(t)\eta(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{4} \langle p_2(t)\sigma_{xx}(t)(\gamma(t), \gamma(t)), \sigma_{xx}(t)(\gamma(t), \gamma(t)) \rangle \\
& + \frac{1}{2} \langle p_2(t)\sigma_{xx}(t)(\gamma(t), \gamma(t)), \delta\sigma(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{2} \langle p_2(t)\delta\sigma(t), \sigma_{xx}(t)(\gamma(t), \gamma(t)) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \langle p_2(t)(\delta\sigma(t) + \delta\sigma_x(t)\gamma(t)), \delta\sigma(t) + \delta\sigma_x(t)\gamma(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{2} \langle p_2(t)\delta\sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)), \delta\sigma(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{2} \langle p_2(t)\delta\sigma(t), \delta\sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle_{\chi_{E_\varepsilon}(t)} \\
\text{(B.5)} \quad & - \langle \mathcal{H}_{xx}(t)\eta(t), \eta(t) \rangle \Big] dt + o(\varepsilon^2) \quad (\varepsilon \rightarrow 0^+),
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left[h_{xxx}(\bar{x}(T))(\gamma(T), \gamma(T), \gamma(T)) \right] = -\mathbb{E} \left[p_3(T)(\gamma(T), \gamma(T), \gamma(T)) \right] \\
& = -\mathbb{E} \int_0^T \left[p_3(t) \left(\frac{1}{2} b_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta b(t) \chi_{E_\varepsilon}(t), \gamma(t), \gamma(t) \right) \right. \\
& \quad + p_3(t) \left(\gamma(t), \frac{1}{2} b_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta b(t) \chi_{E_\varepsilon}(t), \gamma(t) \right) \\
& \quad + p_3(t) \left(\gamma(t), \gamma(t), \frac{1}{2} b_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta b(t) \chi_{E_\varepsilon}(t) \right) \\
& \quad - \frac{3}{2} \langle p_2(t) b_{xx}(t)(\gamma(t), \gamma(t)), \gamma(t) \rangle - \frac{3}{2} \langle p_2(t) \gamma(t), b_{xx}(t)(\gamma(t), \gamma(t)) \rangle \\
& \quad - \frac{3}{2} \langle q_2(t) \sigma_{xx}(t)(\gamma(t), \gamma(t)), \gamma(t) \rangle - \frac{3}{2} \langle q_2(t) \gamma(t), \sigma_{xx}(t)(\gamma(t), \gamma(t)) \rangle \\
& \quad - \frac{3}{2} \langle p_2(t) \sigma_{xx}(t)(\gamma(t), \gamma(t)), \sigma_x(t) \gamma(t) \rangle - \frac{3}{2} \langle p_2(t) \sigma_x(t) \gamma(t), \sigma_{xx}(t)(\gamma(t), \gamma(t)) \rangle \\
& \quad + q_3(t) \left(\frac{1}{2} \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta\sigma(t) \chi_{E_\varepsilon}(t), \gamma(t), \gamma(t) \right) \\
& \quad + q_3(t) \left(\gamma(t), \frac{1}{2} \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta\sigma(t) \chi_{E_\varepsilon}(t), \gamma(t) \right) \\
& \quad + q_3(t) \left(\gamma(t), \gamma(t), \frac{1}{2} \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) + \delta\sigma(t) \chi_{E_\varepsilon}(t) \right) \\
& \quad + \frac{1}{2} p_3(t) \left(\sigma_x(t) \gamma(t), \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)), \gamma(t) \right) \\
& \quad + \frac{1}{2} p_3(t) \left(\sigma_x(t) \gamma(t), \gamma(t), \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) \right) \\
& \quad + \frac{1}{2} p_3(t) \left(\gamma(t), \sigma_x(t) \gamma(t), \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)) \right) \\
& \quad + \frac{1}{2} p_3(t) \left(\sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)), \sigma_x(t) \gamma(t), \gamma(t) \right) \\
& \quad + \frac{1}{2} p_3(t) \left(\sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)), \gamma(t), \sigma_x(t) \gamma(t) \right) \\
& \quad + \frac{1}{2} p_3(t) \left(\gamma(t), \sigma_{xx}(t)(y_1^\varepsilon(t), y_1^\varepsilon(t)), \sigma_x(t) \gamma(t) \right) \Big]
\end{aligned}$$

$$\begin{aligned}
& +p_3(t) \left(\sigma_x(t) \gamma(t), \delta \sigma(t), \gamma(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\sigma_x(t) \gamma(t), \gamma(t), \delta \sigma(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\gamma(t), \sigma_x(t) \gamma(t), \delta \sigma(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\delta \sigma(t), \sigma_x(t) \gamma(t), \gamma(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\delta \sigma(t), \gamma(t), \sigma_x(t) \gamma(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\gamma(t), \delta \sigma(t), \sigma_x(t) \gamma(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\delta \sigma(t), \delta \sigma(t), \gamma(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\delta \sigma(t), \gamma(t), \delta \sigma(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\gamma(t), \delta \sigma(t), \delta \sigma(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\delta \sigma(t), \delta \sigma_x(t) y_1^\varepsilon(t), \gamma(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\delta \sigma(t), \gamma(t), \delta \sigma_x(t) y_1^\varepsilon(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\gamma(t), \delta \sigma(t), \delta \sigma_x(t) y_1^\varepsilon(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\delta \sigma_x(t) y_1^\varepsilon(t), \delta \sigma(t), \gamma(t) \right) \chi_{E_\varepsilon}(t) \\
& +p_3(t) \left(\delta \sigma_x(t) y_1^\varepsilon(t), \gamma(t), \delta \sigma(t) \right) \chi_{E_\varepsilon}(t) + p_3(t) \left(\gamma(t), \delta \sigma_x(t) y_1^\varepsilon(t), \delta \sigma(t) \right) \chi_{E_\varepsilon}(t) \\
& (B.6) - \mathcal{H}_{xxx}(t) (\gamma(t), \gamma(t), \gamma(t)) \Big] dt + o(\varepsilon^2), \quad (\varepsilon \rightarrow 0^+),
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left[h_{xxx}(\bar{x}(T)) (y_1^\varepsilon(T), y_1^\varepsilon(T), y_1^\varepsilon(T), y_1^\varepsilon(T)) \right] \\
& = -\mathbb{E} \left[p_4(T) (y_1^\varepsilon(T), y_1^\varepsilon(T), y_1^\varepsilon(T), y_1^\varepsilon(T)) \right] \\
& = -\mathbb{E} \int_0^T \left[p_4(t) (\delta \sigma(t), \delta \sigma(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \chi_{E_\varepsilon}(t) \right. \\
& \quad + p_4(t) (\delta \sigma(t), y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t)) \chi_{E_\varepsilon}(t) \\
& \quad + p_4(t) (\delta \sigma(t), y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t)) \chi_{E_\varepsilon}(t) \\
& \quad + p_4(t) (y_1^\varepsilon(t), \delta \sigma(t), \delta \sigma(t), y_1^\varepsilon(t)) \chi_{E_\varepsilon}(t) \\
& \quad + p_4(t) (y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t), \delta \sigma(t)) \chi_{E_\varepsilon}(t) \\
& \quad + p_4(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t), \delta \sigma(t)) \chi_{E_\varepsilon}(t) \\
& \quad - 2p_3(t) \left(b_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t), y_1^\varepsilon(t) \right) \\
& \quad - 2p_3(t) \left(y_1^\varepsilon(t), b_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t) \right) \\
& \quad - 2p_3(t) \left(y_1^\varepsilon(t), y_1^\varepsilon(t), b_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \right) \\
& \quad - 2q_3(t) \left(\sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t), y_1^\varepsilon(t) \right) \\
& \quad - 2q_3(t) \left(y_1^\varepsilon(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t) \right) \\
& \quad - 2q_3(t) \left(y_1^\varepsilon(t), y_1^\varepsilon(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \right) \\
& \quad - 2p_3(t) \left(\sigma_x(t) y_1^\varepsilon(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t) \right) \\
& \quad \left. - 2p_3(t) \left(\sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -2p_3(t) \left(y_1^\varepsilon(t), \sigma_x(t) y_1^\varepsilon(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \right) \\
& -2p_3(t) \left(\sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), \sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t) \right) \\
& -2p_3(t) \left(\sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t), \sigma_x(t) y_1^\varepsilon(t) \right) \\
& -2p_3(t) \left(y_1^\varepsilon(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), \sigma_x(t) y_1^\varepsilon(t) \right) \\
& -2 \langle p_2(t) b_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t) \rangle \\
& -2 \langle p_2(t) y_1^\varepsilon(t), b_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\
& -2 \langle p_2(t) \sigma_x(t) y_1^\varepsilon(t), \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\
& -2 \langle p_2(t) \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)), \sigma_x(t) y_1^\varepsilon(t) \rangle \\
& -2 \langle q_2(t) y_1^\varepsilon(t), \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\
& -2 \langle q_2(t) \sigma_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)), y_1^\varepsilon(t) \rangle \\
& -3 \langle p_2(t) \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \\
& -\mathcal{H}_{xxx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), y_1^\varepsilon(t)) \Big] dt + o(\varepsilon^2), \quad (\varepsilon \rightarrow 0^+).
\end{aligned}
\tag{B.7}$$

Substituting (B.4)–(B.7) into the Taylor expansion (3.8), we obtain that

$$\begin{aligned}
& J(u^\varepsilon) - J(\bar{u}) \\
& = -\mathbb{E} \int_0^T \left[-\delta f(t) \chi_{E_\varepsilon}(t) - \delta f_x(t) \gamma(t) \chi_{E_\varepsilon}(t) - \frac{1}{2} \delta f_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \chi_{E_\varepsilon}(t) \right. \\
& \quad + \langle p_1(t), \delta b(t) \rangle \chi_{E_\varepsilon}(t) + \langle q_1(t), \delta \sigma(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \langle p_1(t), \delta b_x(t) \gamma(t) \rangle \chi_{E_\varepsilon}(t) + \langle q_1(t), \delta \sigma_x(t) \gamma(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle p_1(t), \delta b_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle q_1(t), \delta \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle p_2(t) \delta b(t), \gamma(t) \rangle \chi_{E_\varepsilon}(t) + \frac{1}{2} \langle p_2(t) \gamma(t), \delta b(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle q_2(t) \delta \sigma(t), \gamma(t) \rangle \chi_{E_\varepsilon}(t) + \frac{1}{2} \langle q_2(t) \gamma(t), \delta \sigma(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle p_2(t) \delta b_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) + \frac{1}{2} \langle p_2(t) y_1^\varepsilon(t), \delta b_x(t) y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle q_2(t) \delta \sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) + \frac{1}{2} \langle q_2(t) y_1^\varepsilon(t), \delta \sigma_x(t) y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle p_2(t) \sigma_x(t) \gamma(t), \delta \sigma(t) \rangle \chi_{E_\varepsilon}(t) + \frac{1}{2} \langle p_2(t) \delta \sigma(t), \sigma_x(t) \gamma(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle p_2(t) \sigma_x(t) y_1^\varepsilon(t), \delta \sigma_x(t) y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{2} \langle p_2(t) \delta \sigma_x(t) y_1^\varepsilon(t), \sigma_x(t) y_1^\varepsilon(t) \rangle \chi_{E_\varepsilon}(t) \\
& \quad + \frac{1}{4} \langle p_2(t) \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), \delta \sigma(t) \rangle \chi_{E_\varepsilon}(t)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \langle p_2(t) \delta \sigma(t), \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{2} \langle p_2(t) \delta \sigma(t), \delta \sigma(t) \rangle_{\chi_{E_\varepsilon}(t)} + \frac{1}{2} \langle p_2(t) \delta \sigma(t), \delta \sigma_x(t) \gamma(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{2} \langle p_2(t) \delta \sigma_x(t) \gamma(t), \delta \sigma(t) \rangle_{\chi_{E_\varepsilon}(t)} + \frac{1}{2} \langle p_2(t) \delta \sigma_x(t) y_1^\varepsilon(t), \delta \sigma_x(t) y_1^\varepsilon(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{4} \langle p_2(t) \delta \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)), \delta \sigma(t) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{4} \langle p_2(t) \delta \sigma(t), \delta \sigma_{xx}(t) (y_1^\varepsilon(t), y_1^\varepsilon(t)) \rangle_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\delta b(t), y_1^\varepsilon(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (y_1^\varepsilon(t), \delta b(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), \delta b(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} q_3(t) (\delta \sigma(t), y_1^\varepsilon(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} q_3(t) (y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} q_3(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\sigma_x(t) y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (\sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (y_1^\varepsilon(t), \sigma_x(t) y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (\delta \sigma(t), \sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\delta \sigma(t), y_1^\varepsilon(t), \sigma_x(t) y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (y_1^\varepsilon(t), \delta \sigma(t), \sigma_x(t) y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\delta \sigma(t), \delta \sigma(t), \gamma(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (\delta \sigma(t), \gamma(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\gamma(t), \delta \sigma(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\delta \sigma(t), \delta \sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (\delta \sigma(t), y_1^\varepsilon(t), \delta \sigma_x(t) y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (y_1^\varepsilon(t), \delta \sigma(t), \delta \sigma_x(t) y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (\delta \sigma_x(t) y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{6} p_3(t) (\delta \sigma_x(t) y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} + \frac{1}{6} p_3(t) (y_1^\varepsilon(t), \delta \sigma_x(t) y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{24} p_4(t) (\delta \sigma(t), \delta \sigma(t), y_1^\varepsilon(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{24} p_4(t) (\delta \sigma(t), y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{24} p_4(t) (\delta \sigma(t), y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{24} p_4(t) (y_1^\varepsilon(t), \delta \sigma(t), \delta \sigma(t), y_1^\varepsilon(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{24} p_4(t) (y_1^\varepsilon(t), \delta \sigma(t), y_1^\varepsilon(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \\
& + \frac{1}{24} p_4(t) (y_1^\varepsilon(t), y_1^\varepsilon(t), \delta \sigma(t), \delta \sigma(t))_{\chi_{E_\varepsilon}(t)} \Big] dt + o(\varepsilon^2), \quad (\varepsilon \rightarrow 0^+) \\
& = -\mathbb{E} \int_0^T \left[\mathbb{H}(t, \bar{x}(t), u(t)) + \langle \mathbb{S}(t, \bar{x}(t), u(t)), \gamma(t) \rangle \right. \\
& \quad \left. + \frac{1}{2} \langle \mathbb{T}(t, \bar{x}(t), u(t)) y_1^\varepsilon(t), y_1^\varepsilon(t) \rangle \right]_{\chi_{E_\varepsilon}(t)} dt + o(\varepsilon^2), \quad (\varepsilon \rightarrow 0^+).
\end{aligned}$$

This proves (3.11), and completes the proof of Proposition 3.4.

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